

Pure mathematics

2nd Secondary

2nd term

Unit One

"Sequences and series"

- 1- 1** Sequences
- 1 - 2** Series and Summation Notation
- 1 - 3** Arithmetic Sequence.
- 1 - 4** Arithmetic Series.
- 1 - 5** Geometric Sequences.
- 1 - 6** Geometric Series.

Lesson (1): Sequence

Definition

The **sequence** is a function whose domain is the set of the positive integers $\mathbb{Z}^+$ or a subset of it and its range is a set of the real numbers $\mathbb{R}$ where the first term is denoted by T_1 , the second term is denoted by T_2 , and the third term is denoted by T_3 and so on... and the n th term is denoted by T_n the sequence can be expressed by writing down its terms between two brackets as follows:
($T_1, T_2, T_3, \dots, T_n$) or denoted by the symbol (T_n).

Finite and Infinite Sequences

The sequence is **finite** if the number of its terms is finite (i.e. **can be counted**)

The sequence is **infinite** if the number of its terms is infinite (an infinite number of elements)

[1] Show whether each of the following sequences is finite or infinite:

(a) (1, 4, 7, 11,)

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(b) (3, 5, 7, 9, , 21)

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(c) The sequence (T_n) where $T_n = n^2 - 1, n \in \mathbb{Z}^+$

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(d) The sequence (T_n) where $T_n = \frac{2}{n} + 3, n \in \{1, 2, 3, 4, 5\}$

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General term of a Sequence

The general term of a sequence (it is sometimes called n^{th} term) is written as T_n where T_n is the image of the element whose order is n and can be found sometimes through the given terms of the sequence by realizing the relation between the value and order of the term.

[2] Write the first five terms for each of the following sequences whose general term is given by the following rules:

(a) $T_n = n + n^2$

(b) $T_n = \frac{1}{2n - 5}$

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(c) $T_n = \left(\frac{1}{3}\right)^n$

(d) $T_n = \sin\left(\frac{n}{4}\pi\right)$

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(e) $T_n = (-1)^n (n - 2)^2$

(f) $T_n = \frac{(-1)^n}{n^2}$

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[3] Write down the general term for each of the following sequences:

(a) $(2, 5, 8, 11, \dots)$

(b) $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right)$

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(c) $(1, 8, 27, 64, \dots)$

(d) $\left(1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \dots\right)$

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(e) $(-1, 2, -4, 8, -16, \dots)$

(f) $(\cos \frac{\pi}{3}, \cos \frac{2\pi}{3}, \cos \pi, \cos \frac{4\pi}{3}, \dots)$

Increasing and Decreasing Sequences

- The sequence (T_n) is called **increasing** if $T_{n+1} > T_n$
- The sequence (T_n) is called **decreasing** if $T_{n+1} < T_n$
- The sequence (T_n) is called **Constant** if $T_{n+1} = T_n$

[4] Show whether each of the sequences (T_n) is increasing, decreasing or otherwise in each of the following:

(a) $T_n = 3n + 5$

(b) $T_n = \frac{1}{n} + 2$

(c) $T_n = \left(\frac{1}{2}\right)^{n+1}$

(d) $T_n = (-1)^n(n^2 + 1)$

Exercises (1)

[1] Choose the correct answer:

1)	 The real infinite sequence is a function whose domain is (a) $\mathbb{R}$ (b) $\mathbb{R}^+$ (c) $\mathbb{Z}$ (d) $\mathbb{Z}^+$	
2)	The next term in the sequence (1, 2, 4, 7, 11, ...) is (a) 15 (b) 16 (c) 17 (d) 18	
3)	The next term in the sequence $(\frac{3}{13}, \frac{7}{10}, \frac{11}{7}, \frac{15}{4}, \dots)$ is (a) $\frac{19}{2}$ (b) 19 (c) $-\frac{23}{2}$ (d) $\frac{16}{3}$	
4)	The term free of x in the sequence $(8 - 6x, 7 - 4x, 6 - 2x, \dots)$ is (a) T_7 (b) T_6 (c) T_5 (d) T_4	
5)	 In the sequence (T_n) , where $T_{n+1} = n T_n$, $n \geq 1$, if $T_1 = 1$, then $T_2 = \dots$ (a) 1 (b) 2 (c) 4 (d) $\frac{1}{2}$	
6)	The first five terms in the sequence (T_n) in which $T_{n+1} = \frac{(-1)^n}{3 T_n}$, $T_1 = 9$ is (a) $(9, \frac{1}{27}, -9, 27, 9)$ (b) $(9, \frac{1}{9}, -9, \frac{1}{27}, -27)$ (c) $(9, \frac{1}{27}, \frac{-1}{81}, \frac{1}{243}, \frac{-1}{729})$ (d) $(9, \frac{-1}{27}, -9, \frac{1}{27}, 9)$	
7)	In the sequence (T_n) , if $T_1 = 2$, $T_{n+1} = T_n - n$ where $n \geq 1$, then the first five terms are (a) 1, 2, -1, -4, -8 (b) 2, -1, 1, 3, -8 (c) 2, 1, -1, -4, -8 (d) 2, 1, -1, -3, -8	
8)	Given that (T_n) is a sequence in which $T_n = T_1 + T_2 + T_3 + \dots + T_{n-1}$ where $n > 2$, then $T_1 + T_2 + T_3 + T_4 + T_5 - T_6 = \dots$ (a) zero (b) 32 (c) 64 (d) 72	

9)	The sequence is decreasing if T_{n+1} T_n for every $n \geq 1$ (a) > (b) < (c) = (d) $\geq$
10)	If the general term of a sequence (T_n) is $T_n = 3$, then the first five terms in it are (a) (3, 4, 5, 6, 7) (b) (1, 2, 3, 4, 5) (c) (3, 3, 3, 3, 3) (d) (4, 5, 6, 7, 8)
11)	 The sequence whose rule is $T_n = \frac{(-1)^n}{3n-2}$ is called sequence. (a) an increasing (b) a decreasing (c) an alternating (d) a constant
12)	 The sequence whose n^{th} term is $T_n = \frac{2}{n} - 1$, where $n \in \mathbb{Z}^+$ represents (a) an increasing sequence. (b) a decreasing sequence. (c) a constant sequence. (d) an alternating sequence.
13)	 The sequence (3, 5, 7, 9, ..., 21) is (a) decreasing. (b) alternating. (c) finite. (d) infinite.
14)	 The sequence (T_n) = $(n^2 - 1)$, $n \in \mathbb{Z}^+$ is (a) decreasing. (b) alternating. (c) finite. (d) infinite.
15)	The n^{th} term of the sequence (1, 4, 9, 16, ...) is (a) $2n$ (b) $4n$ (c) $2n$ (d) n^2
16)	 The n^{th} term of the sequence (-1, 4, -9, 16, ...) is (a) $-n^2$ (b) $(-1)^n n^2$ (c) $(-n)^2$ (d) $-4n$
17)	 The general term of the sequence $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right)$ is (a) $\frac{n}{2n}$ (b) $\frac{n}{n+1}$ (c) $\frac{1}{n}$ (d) $\frac{n}{2n-1}$

18)	 The general term of the sequence $(1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \dots)$ is $T_n = \dots$ (a) n^{n-1} (b) $(10)^{-n}$ (c) $(\frac{1}{10})^{n-1}$ (d) $(0.1)^{n+1}$
19)	The n^{th} term of the sequence $(2, 2, \frac{8}{3}, 4, \dots)$ is $\dots$ (a) $n - 1$ (b) $2n - 1$ (c) 2^{n-1} (d) $\frac{2^n}{n}$
20)	 The general term of the sequence $((2 \times 3), (3 \times 4), (4 \times 5), (5 \times 6), \dots)$ is $T_n = \dots$ (a) $(n - 1)(n + 1)$ (b) $n(n + 1)$ (c) $2n(n + 1)$ (d) $(n + 1)(n + 2)$
21)	In the sequence $(2, 5, 8, 11, \dots)$ First : The general term is $T_n = \dots$ (a) $2n$ (b) $4n - 2$ (c) $n + 1$ (d) $3n - 1$ Second : $T_{11} = \dots$ (a) 22 (b) 32 (c) 34 (d) 28 Third : The order of the term whose value 59 is $\dots$ (a) T_{15} (b) T_{22} (c) T_{18} (d) T_{20}
22)	In the sequence $(2, 4, 8, 16, \dots)$ First : The general term is $\dots$ (a) $2n$ (b) $n + 1$ (c) $2(n + 1)$ (d) 2^n Second : The order of the term whose value 128 is $\dots$ (a) T_5 (b) T_7 (c) T_8 (d) T_6

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Lesson (2): Series and Summation Notation

The difference between sequence and series

The sequence is an ordered list of numbers

The series is the sum of terms of the sequence.

For example: (2, 5, 8, 11, ...) is a **sequence** while $2 + 5 + 8 + 11 + \dots$ is **the series** correlated with the previous sequence.

The summation notation " Σ " can be used for writing the series in short forms.

Finite series

It is written in the form: $T_1 + T_2 + T_3 + \dots + T_r + \dots + T_n$

where n is a positive integer, T_n is the terms whose order is n in the series and the numerical value of the finite series is called the sum of the terms of the corresponding sequence.

The finite series: $a_1 + a_2 + a_3 + \dots + a_r + \dots + a_n$ can be written in the form $\sum_{r=1}^n (ar)$ and read as (the sum of ar from $r = 1$ to $r = n$.)

Infinite Series

The terms of the infinite series cannot be counted.

For example: the series: $-3 + 9 - 27 + 81 - 243 + \dots$ can be written in the form of $\sum_{r=1}^{\infty} (-3)^r$. The symbol ∞ is used to express infinity.

1) Write down the following series using the summation notation:

[a] $1 + 2 + 3 + 4 + 5 + \dots + 20$

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[b] $2 + 4 + 6 + 8 + 10 + \dots + 60$

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[c] $3 + 6 + 9 + 12 + 15 + 18 + 21$

[d] $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

[e] $1 + 4 + 9 + 16 + \dots + 64$

[f] $\frac{1}{5} + \frac{1}{10} + \frac{1}{15} + \frac{1}{20} + \dots + \frac{1}{50}$

2) Expand each of the following series:

[a] $\sum_{r=1}^5 (3x - 2)$

[b] $\sum_{r=1}^8 ((-1)^r + 4r)$

[c] $\sum_{r=1}^{\infty} \left(\left(\frac{1}{2} \right)^r - 1 \right)$

[d] $\sum_{r=1}^{\infty} \left(\frac{1}{r} - \frac{1}{r+1} \right)$

Algebraic Properties of Summation

If (T_r) and (E_r) are two sequences, $n \in \mathbb{Z}^+$ and $C \in \mathbb{R}$, then:

$$(A) \sum_{r=1}^n C = C n$$

$$(B) \sum_{r=1}^n C T_r = C \sum_{r=1}^n T_r$$

$$(C) \sum_{r=1}^n (T_r + E_r) = \sum_{r=1}^n T_r + \sum_{r=1}^n E_r$$

If you know that $\sum_{r=1}^n r = \frac{n(n+1)}{2}$ and $\sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}$, use the properties of

summation notation $\sum$ to find the value for each of the following:

$$[a] \sum_{r=8}^{12} 2(x+5)$$

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$$[b] \sum_{r=5}^8 (2r^2 - 3r)$$

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Exercises (2)

[1] Choose the correct answer:

1)	$\sum_{r=2}^5 3 = \dots\dots\dots$ (a) 3 (b) 5 (c) 15 (d) 12	
2)	$\sum_{r=1}^4 r = \dots\dots\dots$ (a) 20 (b) 10 (c) 4 (d) 6	
3)	$\sum_{r=1}^{10} r^2 = \dots\dots\dots$ (a) 10 (b) 55 (c) 220 (d) 385	
4)	The value of the series $\sum_{r=7}^{23} 3r$ is $\dots\dots\dots$ (a) 255 (b) 765 (c) 807 (d) 828	
5)	The value of the series $\sum_{r=1}^{15} (r^2 + r + 1)$ is $\dots\dots\dots$ (a) 1375 (b) 3720 (c) 14400 (d) 2232000	
6)	The value of the series $\sum_{n=1}^{10} (-1)^n = \dots\dots\dots$ (a) zero (b) 1 (c) -1 (d) 10	
7)	If $\sum_{r=1}^n (r) = 136$, then $\sum_{r=1}^n (4r) = \dots\dots\dots$ (a) 140 (b) 544 (c) 272 (d) 34	
8)	$\sum_{r=1}^{10} (3r+2) + \sum_{r=11}^{35} (3r+2) = \dots\dots\dots$ (a) $\sum_{r=1}^{35} (3r+2)$ (b) $\sum_{r=1}^{35} (3r+2)^2$ (c) $\sum_{r=1}^{35} (6r+4)$ (d) $\sum_{r=1}^{35} (3r+4)$	

9)	If $\sum_{r=1}^n (r^2) = 55$, $\sum_{r=1}^n (2r - 1) = 25$, then $\sum_{r=1}^n (r^2 + 2r - 1) = \dots\dots\dots$ (a) 55 (b) 30 (c) 80 (d) 1375
10)	The sum of the positive multiples of the number 6 and less than 100 is $\dots\dots\dots$ (a) $\sum_{r=1}^{100} 6r$ (b) $\sum_{r=1}^{16} 6r$ (c) $\sum_{r=1}^{16} r$ (d) $\sum_{r=1}^6 (100 - 6r)$
11)	The twentieth term in the series $(2 \times 4 + 4 \times 6 + 6 \times 8 + \dots)$ equals $\dots\dots\dots$ (a) 1680 (b) 1600 (c) 840 (d) 420
12)	$1 + 2 + 3 + 4 + 5 + \dots + 20$ by using summation notation = $\dots\dots\dots$ (a) $\sum_{r=1}^{20} 1$ (b) $\sum_{r=1}^{20} r^2$ (c) $\sum_{r=1}^{20} (r + 1)$ (d) $\sum_{r=1}^{20} r$
13)	$2 + 4 + 6 + 8 + \dots + 30 = \dots\dots\dots$ (a) $\sum_{r=1}^{30} r$ (b) $\sum_{r=1}^{15} r$ (c) $\sum_{r=1}^{15} 2r$ (d) $\sum_{r=1}^{30} 2r$
14)	$1^2 + 2^2 + 3^2 + 4^2 + \dots + 8^2 = \dots\dots\dots$ (a) $\sum_{r=1}^8 r^2$ (b) $\sum_{r=1}^{64} r$ (c) $\sum_{r=1}^8 2r$ (d) $\sum_{r=1}^8 2r^2$
15)	$3 + 6 + 9 + 12 + \dots = \dots\dots\dots$ (a) $\sum_{r=1}^{\infty} 3r$ (b) $\sum_{r=1}^{\infty} (r + 3)$ (c) $\sum_{r=1}^{\infty} (r + 2)$ (d) $\sum_{r=1}^{\infty} r^3$
16)	 The series $(2 \times 3) + (3 \times 4) + (4 \times 5) + \dots$ by using summation notation = $\dots\dots\dots$ (a) $\sum_{r=1}^{\infty} (r + 1)(r + 2)$ (b) $\sum_{r=2}^{\infty} r(r + 1)$ (c) $\sum_{r=3}^{\infty} r(r - 1)$ (d) All the previous.

17)	 The series $(7 \times 1) + (7 \times 2) + (7 \times 3) + \dots + (7 \times 20)$ by using summation notation = (a) $7 \sum_{r=1}^{140} r$ (b) $\sum_{r=1}^n 7r$ (c) $\sum_{r=1}^{20} 7 \cdot \sum_{r=1}^{20} r$ (d) $\sum_{r=1}^{20} 7r$
18)	 $-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$ by using summation notation = (a) $\sum_{r=1}^{\infty} -1 \cdot \sum_{r=1}^{\infty} 2^{-r}$ (b) $\sum_{r=0}^{16} (-1)^r \left(\frac{1}{2}\right)^{r+1}$ (c) $\sum_{r=1}^{\infty} 2^{-r}$ (d) $\sum_{r=1}^{\infty} \left(-\frac{1}{2}\right)^r$
19)	$(1 \times 2 \times 3) + (3 \times 4 \times 5) + (5 \times 6 \times 7) + \dots$ by using summation notation equals (a) $\sum_{r=1}^{\infty} r(r+1)(r+2)$ (b) $\sum_{r=1}^7 r(r+1)(r+2)$ (c) $\sum_{r=1}^5 (2r-1)(2r)(2r+1)$ (d) $\sum_{r=1}^{\infty} (2r-1)(2r)(2r+1)$
20)	$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots + \frac{1}{20}$ by using summation notation = (a) $\sum_{r=1}^{20} \frac{1}{2r}$ (b) $\sum_{r=1}^{\infty} \frac{1}{r+1}$ (c) $\sum_{r=1}^{10} \frac{1}{2r}$ (d) $\sum_{r=0}^{20} \frac{1}{r+2}$

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Lesson (3) Arithmetic Sequence

The arithmetic sequence is the sequence in which the difference between a term and the directly previous term to it equals a constant and it is called the **common difference** of the sequence. it is denoted by the symbol (**d**)

$$d = T_{n+1} - T_n \text{ for each } n \in \mathbb{Z}^+$$

[1] Determine which of the following sequences are arithmetic and which are non-arithmetic, then find the common difference in case the sequence is arithmetic:

[1] (34 , 30 , 26 , 22 , 18)

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[2] (7 , 12 , 17 , 22 , 27)

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[3] (-12 , -18 , - 24 , - 30 , - 36)

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[4] (7 , 7 , 7 , 7 , 7)

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Types of arithmetic sequences

The sequence is **increasing** if $d > 0$

The sequence is **decreasing** if $d < 0$

The n^{th} Term of the arithmetic sequence

The n^{th} term of the arithmetic sequence (T_n) whose first term is **a** and common Difference is **d** can be deduced as follows **$T_n = a + (n - 1) d$**

Answer the following questions:

(1) Find the number of the terms of the arithmetic sequence (2 , 5 , 8 , ... ,80).

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(2) In the arithmetic sequence (63 , 59 , 55 , , - 133) find :

(a) The value of the seventh term

(b) the number of the sequence terms

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(3) Find the order and value of the first negative term in the arithmetic sequence (67, 64 , 61 ,)

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(4) Find the order and value of the first term whose value is greater than 180 in the arithmetic sequence

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(5) Write down the first three terms of the sequence $(T_n) = (2 + 5n)$, then find the Order of the term whose value is 72 of the sequence and find the order of the First term whose value is more than 100

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(6) What is the value of n in the arithmetic sequence whose first term = 3 ,
 $T_n = 39$ and $T_{2n} = 79$ find the sequence.

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(7) Find the arithmetic sequence whose fifth term = 21 , and the tenth term = 3 times the second term .

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(8) (T_n) is an arithmetic sequence in which $T_1 + T_2 = 9$ and $T_5 = 22$, find this sequence.

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(9) Find the arithmetic sequence whose sixth term = 20 and the ratio between the fourth and tenth terms is 4: 7.

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(10) Find the arithmetic sequence whose fourth term = 11 and the sum of the fifth and ninth terms= 40, then find the order of the term whose value is 152 in this sequence.

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(11) Find the arithmetic sequence in which the arithmetic mean between its third and seventh terms is 19 and its tenth term is greater than twice of its fourth term by 2.

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(12) (T_n) is an arithmetic sequence in which: $T_2 + T_4 = 42$ and $T_3 \times T_5 = 315$, find this sequence.

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(13) If $(8, a, \dots, b, 68)$ form an arithmetic sequence of sixteen terms, find the values of a and b .

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(14) If $36, a, 24, b$ are consecutive terms of an arithmetic sequence, find the value of a and b .

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Arithmetic means

If a, b and c are three consecutive terms in an arithmetic sequence, then b is called the arithmetic mean between the two terms a and c where $b - a = c - b$,

$$2b = a + c, \text{ then } b = \frac{a + c}{2}$$

(15) If the arithmetic mean between a and b is 8 and the arithmetic mean between $4a$ and $2b$ is 20, find the value of a and b

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(16) Insert 16 arithmetic means between 27 and -24

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(17) Find the arithmetic sequence whose ninth term = 25 and the arithmetic mean between the third and fifth terms is 10

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(18) Find the number of the arithmetic means are inserted between 1 and 17 and the seventh mean equals three times the second mean

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Exercises (3)

[1] Choose the correct answer:

1)	 All the following sequences can be arithmetic except (a) $(3, 7, 11, 15, \dots)$ (b) $(-11, -15, -19, -23, \dots)$ (c) $(\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots)$ (d) $(\frac{21}{5}, \frac{16}{5}, \frac{11}{5}, \frac{6}{5}, \dots)$	
2)	All the following sequences are arithmetic except (a) $(2n - 5)$ (b) $(43 - 7n)$ (c) (2^n) (d) $(3n)$	
3)	 From the following sequences , the arithmetic sequence is (a) $(T_n) = (\frac{n+1}{n})$ (b) $(T_n) = ((n+1)^2)$ (c) $(T_n) = (\frac{3}{n}(n+2))$ (d) $(T_n) = (\frac{n^3-1}{n^2+n+1})$	

4)	The sequence (T_n) is an arithmetic sequence if and only if for $n > 1$ (a) $T_n + T_{n+1} = \text{constant}$. (b) $\frac{T_{n+1}}{T_n} = \text{constant}$. (c) $T_{n-1} + T_{n+1} = 2 T_n$ (d) $T_{n+1} - T_n = \frac{1}{n}$
5)	The general term of the sequence $(a, a + d, a + 2d, \dots)$ is (a) $d n + (d - a)$ (b) $a + n d$ (c) $a + (n + 1) d$ (d) $d n + (a - d)$
6)	The n^{th} term of the arithmetic sequence $(3, 5, 7, \dots)$ equals (a) $3 n$ (b) $2 n + 1$ (c) $5 n - 2$ (d) 2^n
7)	If $(29, X, \dots, 3 X, 95)$ is an arithmetic sequence, then $X = \dots$ (a) 30 (b) 31 (c) 32 (d) 33
8)	The number of terms of the sequence $(2, 8, 14, \dots, 68)$ equals terms. (a) 6 (b) 8 (c) 12 (d) 16
9)	The last term in the arithmetic sequence whose first term = 3 and its common difference = 5 and has 25 terms equals (a) 113 (b) 118 (c) 123 (d) 128
10)	The value of the middle term of the sequence $(2, 5, 8, 11, \dots, 128)$ is (a) 22 (b) 43 (c) 65 (d) 96
11)	 In the arithmetic sequence $(12, 14, 16, \dots)$, the order of the term whose value 102 is (a) T_{26} (b) T_{48} (c) T_{46} (d) T_{45}
12)	In the arithmetic sequence $(22, 20, 18, \dots)$, the order of the term whose value equals zero is (a) 8 (b) 10 (c) 12 (d) 14

13)	The value of T_{15} from the end in the arithmetic sequence $(19, 15, 11, \dots, -61)$ equals	(a) -8	(b) -5	(c) -1	(d) -37
14)	The number of terms in an arithmetic sequence is 20, then the fourth term from the end is the term from the beginning.	(a) 15	(b) 16	(c) 17	(d) 18
15)	The order of the first term smaller than -180 in the arithmetic sequence $(64, 61, 58, \dots)$ is	(a) 81	(b) 82	(c) 83	(d) 84
17)	The value of the first term greater than 1000 from the arithmetic sequence $(2, 9, 16, \dots)$ is	(a) 1003	(b) 1004	(c) 1005	(d) 1006
18)	The order of the first positive term in the arithmetic sequence $(-48, -45, -42, \dots)$ is	(a) T_{16}	(b) T_{17}	(c) T_{18}	(d) T_{19}
19)	The order of the last positive term in the sequence $(28, 25, 22, \dots)$ is	(a) T_9	(b) T_{10}	(c) T_{11}	(d) T_{12}
20)	If (T_n) is an arithmetic sequence, $T_3 - T_5 = 6$, $T_4 = 16$, then the value of the first negative term in the sequence equals	(a) -4	(b) -3	(c) -2	(d) -1
21)	An arithmetic sequence whose first term is 5, $T_{n+1} = T_n + 3$, then its fifth term equals	(a) 12	(b) 20	(c) 17	(d) 19
22)	If (T_n) is an arithmetic sequence, $T_{13} + T_{18} = 68$, then $T_{14} + T_{17} = \dots\dots\dots$	(a) 68	(b) 136	(c) 34	(d) 204

23)	 The arithmetic mean of the two numbers 8 and 12 is	
	(a) 8 (b) 12 (c) 20 (d) 10	
24)	 If the arithmetic mean of the two numbers X and 26 is 21 , then $X =$	
	(a) 26 (b) 16 (c) 42 (d) 21	
25)	In an arithmetic sequence $(T_n) : \frac{T_{n-1} + T_{n+1}}{T_n} =$	
	(a) T_{n+2} (b) T_{2n} (c) zero (d) 2	
25)	In an arithmetic sequence , the fifth mean is the term.	
	(a) fifth (b) fourth (c) tenth (d) sixth	
26)	 If a and b are two arithmetic means between X and y , then $\frac{y-X}{b-a} =$	
	(a) 2 (b) 3 (c) 4 (d) 6	
27)	If you insert 12 arithmetic sequence between -14 and 51 , then the seventh mean =	
	(a) 21 (b) 16 (c) 26 (d) 31	
28)	The difference between two numbers is 5 and their arithmetic mean is 6.5 , then the two numbers are	
	(a) 5 , 10 (b) -1 , 4 (c) 2 , 7 (d) 9 , 4	
29)	 If (T_n) is an arithmetic sequence where $T_n = 3n + 2$, then the arithmetic mean between T_5 and T_{11} equals	
	(a) 8 (b) 16 (c) 22 (d) 26	

Lesson (4): Arithmetic Series

The Sum of the First n terms of Arithmetic Sequence:

First: the sum of n^{th} terms of an arithmetic sequence in

terms of its first term (a) and the last term (ℓ) is : $S_n = \frac{n}{2}(a + \ell)$

Second: Finding the sum of n^{th} terms of an arithmetic sequence in terms of its first term

and its common difference: $S_n = \frac{n}{2}(2a + (n - 1) d)$

1 Find the sum of the first ten terms of the arithmetic sequence (14 , 18 , 22 , ...).

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2 Find the sum of the first thirty terms of the sequence (T_n) where $T_n = 2n + 3$

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3 Find the sum of the terms of the arithmetic sequence (2 , 5 , 8 , ... , 80).

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4 Find the number of the terms necessary to be taken from the sequence (16 , 20 , 24 , ...) starting from the first term to get a sum equal to 456

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5 Find the number of the terms necessary to be taken from the sequence
(27 , 24, 21 , ...) starting from the first term to vanish the sum.

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6 If the sum of n-terms from an arithmetic sequence is identified by the rule:
 $S_n = 2n(7 - n)$, find:

(a) T_7

(b) The number of terms necessary to be taken from the sequence starting
from the first term to make the sum equal to - 240

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7 Find the number of the terms necessary to be taken from the sequence
(27 , 24, 21 , ...) starting from the first term to vanish the sum.

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8 Find the least number of terms that can be taken from the sequence
(89 , 81 , 73 , ...) starting from the first term to make the negative sum.

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9 Find the greatest number of terms that can be taken from the sequence
(25 , 21 , 17 , ...) starting from the first term to make the positive sum

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10 In the arithmetic sequence (5 , 8 , 11 , ...) ,find:

- (a) The sum of its first twenty terms.
- (b) The sum of ten terms starting from the seventh term.
- (c) The sum of the sequence terms starting from T_{10} up to T_{20}

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11 In the sequence $(T_n) = (32 , 28 , 24 , \dots)$.

- (a) Find the order and value of its first negative term.
- (b) Find the number of terms which makes the sum greater than zero.

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12 In the arithmetic sequence (25 , 23 , 21 , ...) , find:

- (a) The greatest sum of the sequence.
 - (b) The number of terms whose sum = 120 starting from the first term "
- Explain the existence of two answers".

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13 Find the arithmetic sequence (T_n) whose second term = 13 , and the sum of its first ten terms = 235 .

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14 An arithmetic sequence in which $T_{36} = 0$. If the sum of the first n terms = twice the sum of its first five terms, find the value of n .

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15 If n arithmetic means are inserted between 1 and 31 and the ratio of the seventh mean to the last mean is $\frac{15}{29}$ find n and sum of the sequence?

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16 Find the arithmetic sequence in which $T_4 = 24$ and the ratio between the sum of the first five terms to the sum of the next five terms directly is 1: 2

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17 What is the arithmetic sequence whose first term is greater than twice its fifth term by 2 and the arithmetic mean of the third and sixth terms is 16?

How many terms should be taken starting from the first term to make the sum equal to zero?

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WITH SHERIF SAAD

Exercises (4)

[1] Choose the correct answer:

1)	 The value of the arithmetic series $\sum_{r=1}^5 (2r + 1)$ equals (a) 25 (b) 30 (c) 35 (d) 40	
2)	 The value of the series $4 + 9 + 14 + \dots + (5n - 1)$ by using the summation notation equals (a) $\sum_{r=4}^n (5r - 1)$ (b) $\sum_{r=1}^n (5r + 1)$ (c) $\sum_{r=1}^n (5r - 1)$ (d) $\sum_{r=1}^{5n-1} (3r + 1)$	
3)	 The value of the series $7 + 12 + 17 + 22$ by using the summation notation equals (a) $\sum_{r=1}^4 (5r + 2)$ (b) $\sum_{r=1}^4 (4r + 3)$ (c) $\sum_{r=1}^4 (7r + 1)$ (d) $\sum_{r=1}^4 (3r + 4)$	
4)	 The sum of the terms of the arithmetic sequence $(3, 5, 7, \dots, (2n + 1))$ starting from its first term equals (a) $n(n + 1)$ (b) $n(n + 2)$ (c) $n(n + 5)$ (d) $n(n + 2)(n + 3)$	
5)	The sum of the first term and last term of an arithmetic sequence is 46 and the sum of all its terms is 345 , then the number of its terms is (a) 25 (b) 15 (c) 23 (d) 22	
6)	The first term of an arithmetic sequence is 12 and its last term is -26 and the sum of its terms equals -140 , then the sequence is (a) $(12, 8, 4, \dots, -26)$ (b) $(12, 9, 6, \dots, 26)$ (c) $(12, 6, 0, \dots, -26)$ (d) $(12, 10, 8, \dots, -26)$	
7)	The sum of terms in the arithmetic series $89 + 85 + 81 + \dots + 33 = \dots$ (a) 900 (b) 910 (c) 895 (d) 915	

8)	 The sum of terms in the arithmetic series $2 + 5 + 8 + \dots + 62 = \dots\dots\dots$ (a) 664 (b) 670 (c) 660 (d) 672
9)	 The sum of the first 30 terms from the sequence (T_n) where $T_n = 2n + 3$ is $\dots\dots\dots$ (a) 1000 (b) 1024 (c) 1020 (d) 1010
10)	 The sum of natural numbers divisible by 3 between 30 and 50 equals $\dots\dots\dots$ (a) 81 (b) 243 (c) 343 (d) 512
11)	 The arithmetic sequence whose second term is 13 and the sum of its first 10 terms equals 235 is $\dots\dots\dots$ (a) (8 , 13 , 18 , ...) (b) (9 , 13 , 17 , ...) (c) (12 , 13 , 14 , ...) (d) (10 , 13 , 16 , ...)
12)	The first term of an arithmetic sequence is 7 and the sum of its first 10 terms = 250 , then $T_{20} = \dots\dots\dots$ (a) 87 (b) 83 (c) 86 (d) 91
13)	 The number of the terms necessary to be taken from the sequence (27 , 24 , 21 , ...) starting from the first term to vanish the sum is $\dots\dots\dots$ terms. (a) 18 (b) 19 (c) 20 (d) 21
14)	If (T_n) is an arithmetic sequence , the number of its terms is (n) , $T_1 = 12$, $T_n = 78$, $S_n = 1035$, then $T_4 = \dots\dots\dots$ (a) 15 (b) 18 (c) 21 (d) 24
15)	 The greatest number of terms can be taken from the sequence $(T_n) = (32 , 28 , 24 , \dots)$ starting from the first term to get a positive sum is $\dots\dots\dots$ (a) 16 (b) 17 (c) 18 (d) 20
16)	 In the arithmetic sequence (5 , 8 , 11 , ...) , then : First : The sum of its first twenty terms = $\dots\dots\dots$ (a) 670 (b) 820 (c) 525 (d) 690 Second : The sum of ten terms starting from the seventh term = $\dots\dots\dots$ (a) 375 (b) 385 (c) 355 (d) 365 Third : The sum of the sequence terms starting from T_{10} up to $T_{20} = \dots\dots\dots$ (a) 513 (b) 510 (c) 517 (d) 520

Lesson (5): Geometric Sequences

The sequence (T_n) where $T_n \neq 0$ is called a geometric sequence if $\frac{T_{n+1}}{T_n} = a$ constant for each $n \in \mathbb{Z}^+$, The constant is called the **common ratio** of the sequence and is denoted by the symbol **(r)**

The n^{th} Term of the geometric sequence

The n^{th} term of the geometric sequence (T_n) whose first term is a and common ratio is r can be deduced as follows: $T_n = a r^{n-1}$

Answer the following questions:

1 If (T_n) is a sequence where $T_n = 5 \times 2^n$, prove that it is a geometric sequence, then find its first three terms

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2 In the geometric sequence $(\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, -1, \dots)$, find:

(a) Its tenth term.

(b) The order of the term whose value = 1024

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3 Find the geometric sequence whose common ratio = $\frac{1}{2}$ and its third term = 24 .

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4 Find the geometric sequence in which $T_3 = 12$ and $T_8 = 384$.

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5 Show that the sequence (T_n) where: $T_n = \frac{3}{8}(2)^n$ is a geometric sequence, then find its eighth term and the order of the term whose value is 768

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Remark:

Types of geometric sequences

The sequence is **increasing** if $r > 1, a > 0$ or $0 < r < 1, a < 0$

The sequence is **decreasing** if $0 < r < 1, a > 0$ or $r > 1, a < 0$

Geometric means

If a, b and c are three successive terms of a geometric sequence, then b is known as the geometric mean

between the two numbers a and c where: $\frac{a}{b} = \frac{b}{c} \rightarrow b^2 = ac$, then $b = \pm \sqrt{ac}$

6 Find the geometric mean between 16 and 49.

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7 Find the two numbers whose arithmetic mean is 5 and their geometric mean is 3.

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8 Find two positive numbers whose positive geometric mean is greater than one of those two numbers by 2 and is less than the other number by 3.

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9 Insert five positive geometric means between $\frac{8}{27}$, $\frac{27}{8}$

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10 If several geometric means are inserted between 2 and 1458 and the ratio between the sum of the first two means to the sum of the last two means is 1:27, find the number of these means.

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11 Find the geometric sequence whose all terms are positive, the first term is four times the third term and the sum of the second and fifth terms = 36 .

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WITH SHERIF SAAD

Exercises (5)

[1] Choose the correct answer:

1)	 The fifth term from the sequence (T_n) when $T_n = 2 \times (3)^{n-1}$ equals (a) 81 (b) 162 (c) 324 (d) 243
2)	In the geometric mean $(T_n) = (-2560, 1280, \dots, -10)$, then the fourth term from the end = (a) 40 (b) -40 (c) 80 (d) -80
3)	Which of the following is a geometric sequence ? (a) $(2, 5, 8, 11, \dots)$ (b) $(3, 33, 3, 33, \dots)$ (c) $(5, -5, 5, -5, \dots)$ (d) $(1, 4, 8, 16, 32, \dots)$
4)	 All the following sequences are geometric except (a) $(3, -6, 12, -24, \dots)$ (b) $(\log a, \log a^2, \log a^3, \log a^4, \dots)$ (c) $\left(\frac{3}{2}, 1, \frac{2}{3}, \frac{4}{9}\right)$ (d) $\left(\frac{3b}{2a}, 3, \frac{6a}{b}, \frac{12a^2}{b^2}\right)$
5)	 Of all the following sequences, the geometric sequence is (a) $(T_n) = (4n^2)$ for each $n \geq 1$ (b) $(T_n) = \left(\frac{1}{4} \times T_{n-1}\right)$ for each $n \geq 2, T_1 = 1$ (c) $(T_n) = (2^n - 1)$ for each $n \geq 1$ (d) $(T_n) = (\log(3 \times 2^n))$ for each $n \geq 1$
6)	 The common ratio of a geometric sequence = $\frac{1}{2}$ and its third term is 24, then the sequence is (a) $(48, 24, 12, \dots)$ (b) $(6, 12, 24, \dots)$ (c) $(96, 48, 24, \dots)$ (d) $\left(\frac{8}{3}, 8, 24, \dots\right)$
7)	The first term in a geometric sequence is 2 and its sixth term = 64, then the sequence is (a) $(2, 8, 32, \dots)$ (b) $(2, 4, 8, \dots)$ (c) $(2, -4, 8, \dots)$ (d) $(2, 6, 18, \dots)$

8)	The sequence $\left(\frac{1}{243}, -\frac{1}{81}, \frac{1}{27}, -\frac{1}{9}, \frac{1}{3}\right)$ is (1) a finite sequence. (2) an increasing sequence. (3) an alternating sequence. (4) a geometric sequence , its common ratio is -3 (a) (1) only. (b) (1) and (2) only. (c) (3) and (4) only. (d) (1) , (3) and (4) only.
9)	The n^{th} term in the geometric sequence $(3, -6, 12, \dots)$ is (a) $3(-2)^{n-1}$ (b) $3(-2)^n$ (c) $(-6)^{n-1}$ (d) $(-6)^n$
10)	If $(-1, -5, -25, -125, \dots)$ is a geometric sequence , then for each $n > 1$ (a) $T_n = \frac{1}{5} T_{n-1}$ (b) $T_n = -5 T_{n-1}$ (c) $T_n = 10 T_{n-1}$ (d) $T_n = 5 T_{n-1}$
11)	 The next term of the geometric sequence $(8, 6, \frac{9}{2}, \frac{27}{8}, \dots)$ is (a) $\frac{11}{8}$ (b) $\frac{27}{16}$ (c) $\frac{9}{4}$ (d) $\frac{81}{32}$
12)	If a, b, c, d, e form a geometric sequence , then $\frac{e}{c} = \dots\dots\dots$ (a) $\frac{d}{c}$ (b) $\frac{b}{a}$ (c) $\frac{c}{b}$ (d) $\frac{d}{b}$
13)	The geometric sequence whose first term = a and its common ratio = r is increasing if (a) $a > 0, -1 < r < 0$ (b) $a > 0, 0 < r < 1$ (c) $a < 0, -1 < r < 0$ (d) $a < 0, 0 < r < 1$
14)	The geometric sequence whose first term = a and its common ratio = r is decreasing if (a) $a > 0, -1 < r < 0$ (b) $a > 0, 0 < r < 1$ (c) $a < 0, -1 < r < 0$ (d) $a < 0, 0 < r < 1$
15)	The geometric sequence whose first term = a and its common ratio = r is not alternating sign if (a) $a > 0, r < -1$ (b) $a > 0, -1 < r < 0$ (c) $a > 0, 0 < r < 1$ (d) $a < 0, -1 < r < 0$

16)	The order of the term whose value $\frac{1}{243}$ from the geometric sequence (81 , 27 , 9 , ...) equals	(a) 5	(b) 7	(c) 9	(d) 10
17)	The order of the first term less than one in the geometric sequence (1024 , 512 , 256 , ...) is	(a) 7	(b) 10	(c) 12	(d) 14
18)	If the third term in a geometric sequence = 4 , then the product of the first five terms is	(a) 4^2	(b) 4^3	(c) 4^5	(d) 4^6
19)	If the third term from a geometric sequence equals square its first term , and its second term = 8 , then its sixth term =	(a) 120	(b) 124	(c) 128	(d) 132
20)	 All terms of a geometric sequence are positive , $T_3 + T_4 = 6 T_2$, $T_7 = 320$, then the sequence is	(a) (2 , 10 , 20 , ...)	(b) (24 , 16 , 8 , ...)	(c) (3 , 15 , 75 , ...)	(d) (5 , 10 , 20 , ...)
21)	If a , b , c form a geometric sequence , then	(a) $bc = a^2$	(b) $ab = c^2$	(c) $b^2 = ac$	(d) $abc = 1$
22)	The geometric mean of the two numbers 4 , 16 is	(a) 10	(b) 64	(c) 8	(d) ± 8
23)	 If (T_n) is a geometric sequence where $T_n = 7 \times (3)^{n-1}$, then the geometric mean between T_3 and T_7 is	(a) ± 570	(b) ± 567	(c) ± 540	(d) ± 560

24)	Which of the following is a geometric mean of the two quantities a^4 and b^{16} ? (a) $a^4 b^{16}$ (b) $a^2 b^8$ (c) $a^2 b^4$ (d) $a b^4$
25)	In any geometric sequence , $T_1 \times T_7 = \dots\dots\dots$ (a) $(T_1)^2$ (b) $(T_4)^2$ (c) $(T_7)^2$ (d) $(T_8)^2$
26)	If (T_n) is a geometric sequence whose first term = a and its common ratio = a , then its 5 th term = (a) a^3 (b) a^4 (c) a^5 (d) a^6
27)	In a geometric sequence of positive terms , if the 3 rd term equals the multiplicative inverse of the first term , then the second term = (a) zero (b) - 1 (c) 1 (d) 2
28)	If the arithmetic mean of a and b equals 9 , the arithmetic mean of $\frac{1}{a}$, $\frac{1}{b}$ equals $\frac{1}{4}$, then the positive geometric mean of a , b equals (a) 2 (b) 3 (c) 6 (d) 6.5

Lesson (6): Geometric Series

Sum of the First n terms of a Geometric Series

First: To find the sum of n terms of a geometric series in terms of its first term and common ratio: $S_n = \frac{a(1 - r^n)}{1 - r}$ where $r \neq 1$

Second: to find the n terms of a geometric series in terms of its first and last terms

$$S_n = \frac{a - lr}{1 - r} \text{ where } r \neq 1.$$

If $r = 1$, then $S_n = a + a + a + \dots + a$ (to n terms) $= n a$ *i.e.* $S_n = \sum_{i=1}^n a = n a$

Answer the following questions:

1 Find the sum for each of the following geometric sequences:

(a) (6, 12, 24, ... to 6 terms)

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(b) (125, 25, 5, to 6 terms)

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(c) (3, -6, 12,, 768)

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Sum of Infinite geometric series

The n^{th} term of the geometric sequence (T_n) whose first term is a and common

ratio is r can be deduced as follows: $S_{\infty} = \frac{a}{1-r}$, $|r| < 1$. *i.e.* $-1 < r < 1$

2 Which of the following geometric sequences can be summed up to ∞ , then find the sum if possible:

(a) (24 , 12 , 6 ,)

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(b) (3 , - 6 , 12 ,)

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(c) $\left(\frac{1}{32} , \frac{1}{16} , \frac{1}{8} , \dots\dots\right)$

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(d) $(2 \times 5^{1-n})$

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3 Find the sum of an infinite number of terms for each of the following geometric sequence:

(a) $(3 , \sqrt{3} , 1 , \dots)$

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(b) $(T_n) = (3^{3-n})$

4 Put each of the following recurring decimals in the form of common fractions using the infinite geometric sequence: $0.\overline{7}$, $0.2\overline{4}$, $0.8\overline{63}$

5 Find the geometric sequence whose first term = 243 , last term = 1 , and the sum of its terms = 364

6 Find the geometric sequence whose sum is 1093, last term is 729 and its common ratio is 3

7 How many terms are to be taken from the geometric sequence $(3, 6, 12, \dots)$ starting from its first term to make the sum of these terms = 381?

8 Prove that the sequence $(T_n) = (10 \times 2^{n-2})$ is a geometric sequence, then find the number of its terms whose sum is 2555 starting from the first term.

9 Find the number of the terms of the geometric sequence whose sum of its terms is $121\frac{4}{9}$ and its first term equals 18 and the last term equals $\frac{1}{9}$

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10 (T_n) is a geometric sequence whose terms are positive in which $T_2 = 6$ and $T_3 - T_1 = 9$. Find the sequence and sum of the first twelve terms.

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11 Find the geometric sequence whose sum of its first five terms = 7.75 and the sum of the consecutive five terms = 248 .

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12 The first term of a geometric sequence of an infinite number of terms = 18 and the fourth term = $\frac{16}{3}$. What is the sum of this sequence?

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13 If the sum of an infinite geometric sequence is $\frac{375}{4}$, and the sum of its first and second terms is 90, prove that there are two sequences and find them .

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14 Find the geometric sequence whose sum of its first and second terms = 16 , and the sum of an infinite number of its terms = 25.

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Exercises (6)

[1] Choose the correct answer:

1)	The sum of the geometric series in which $a = \frac{1}{2}$, $r = -2$, $n = 10$ equals (a) -170.5 (b) 158 (c) 164 (d) -164
2)	If a geometric sequence whose first term $a = 2$ and its common ratio $r = 1$, then the sum of the first 10 terms of it = (a) 20 (b) 2 (c) 10 (d) 1024
3)	 The sum of the geometric series in which $a = 9$, $r = 3$, $l = 6561$ is (a) 29524 (b) 9837 (c) 2954 (d) 8937
4)	The sum of the first 8 terms of the geometric series $\frac{1}{4} + \frac{1}{2} + 1 + 2 + \dots$ is (a) $63 \frac{3}{4}$ (b) 32 (c) $31 \frac{3}{4}$ (d) 64
5)	$3 + 6 + 12 + \dots + 192 = \dots$ (a) 192 (b) 381 (c) 189 (d) 765
6)	 The sum of the geometric sequence $(3, -6, 12, \dots, 768)$ equals (a) -98 (b) -314 (c) 498 (d) 513
7)	The sum of the first eight terms of the geometric sequence (T_n) where $T_n = 2(3)^{n+1}$ equals (a) 56014 (b) 58094 (c) 59040 (d) 4950
8)	The sum of five terms of the geometric sequence $(1, 3, 9, \dots)$ starting from its third term equals (a) 1089 (b) 2013 (c) 998 (d) 1054

9)	 A geometric sequence in which the sum of the first n terms is given by the relation $S_n = 3^{n+1} - 4$, then the third term equals (a) 18 (b) 23 (c) 54 (d) 77
10)	 The number of terms must be taken from the geometric sequence (3 , 6 , 12 , ...) starting from its first term to make the sum of these terms = 381 is (a) 8 (b) 6 (c) 9 (d) 7
11)	 The geometric sequence whose first term = 243 , last term = 1 , and the sum of its terms = 364 is (a) (243 , 27 , 3 , ...) (b) (729 , 243 , ... , 1) (c) (243 , 81 , ... , 1) (d) (243 , 121.5 , 60.75 , ... , 1)
12)	 The geometric sequence whose sum is 1093 , last term is 729 and its common ratio is 3 is (a) (1 , 3 , 9 , ... , 729) (b) (2 , 6 , 18 , ... , 729) (c) (3 , 9 , 27 , ... , 729) (d) (-3 , 9 , -27 , ... , 729)
13)	A geometric sequence , the sum of its fourth term and sixth term = 120 and the sum of its fifth term and seventh term = 240 , then the sum of the first 10 terms from the sequence = (a) 720 (b) 1023 (c) 3069 (d) 6138
14)	The number of terms of a geometric sequence is $(2n)$ and its common ratio is (r) , then ratio between the sum of its odd-ordered terms to the even-ordered terms equals (a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$ (c) r^2 (d) $\frac{n}{r}$
15)	The terms of a geometric sequence are positive and different , the sum of the first twelve terms equals 273 times the sum of its first four terms , then the common ratio = (a) ± 4 (b) ± 2 (c) 17 (d) 2

16)	We can find the sum of an infinite number of terms of a geometric sequence if and only if	(a) $r < 1$	(b) $r > 1$	(c) $ r < 1$	(d) $ r > 1$
17)	The sum of an infinite number of terms of the geometric sequence $(25, -5, 1, \dots)$ equals	(a) 22	(b) 21	(c) $20 \frac{5}{6}$	(d) $21 \frac{3}{4}$
18)	 The sum of an infinite number of terms of the geometric sequence $(3, \sqrt{3}, 1, \dots)$ equals	(a) $\frac{5 + 2\sqrt{3}}{2}$	(b) $\frac{9 + 3\sqrt{3}}{2}$	(c) $\frac{8 + 4\sqrt{3}}{3}$	(d) $3 + \sqrt{3}$
19)	 The sum of infinite number of terms of the geometric sequence $(T_n) = (3)^{3-n}$ equals	(a) 13	(b) $13 \frac{1}{2}$	(c) $13 \frac{1}{3}$	(d) $12 \frac{1}{2}$
20)	$\sum_{i=1}^{\infty} 4 \left(\frac{1}{2}\right)^{i-1} = \dots\dots\dots$	(a) 8	(b) 4	(c) $4 \frac{1}{2}$	(d) 2
21)	$\sum_{r=1}^{\infty} 27 (3)^{1-r} = \dots\dots\dots$	(a) 40	(b) $\frac{81}{5}$	(c) 39	(d) $\frac{81}{2}$
22)	 $\overline{0.432} = \dots\dots\dots$	(a) $\frac{214}{495}$	(b) $\frac{389}{900}$	(c) $\frac{43}{99}$	(d) $\frac{16}{37}$
23)	 If the sum of an infinite number of terms of a geometric sequence whose first term is 12 is 96, then its common ratio equals	(a) $\frac{1}{3}$	(b) $\frac{1}{2}$	(c) $\frac{7}{8}$	(d) $\frac{3}{4}$

Unit Two

Permutations, Combinations

2 - 1: Counting Principle.- Permutations.

2 - 2: Combinations.

Lesson (1): Counting principle - Permutations

Fundamental Counting Principle

If the number of ways to perform a certain task equals n way and the number of ways to perform another task equals m way, then the number of ways to perform the first and second tasks together = $m \times n$ ways.

Answer the following questions:

1 How many three -digit numbers can be formed from the elements $\{2, 3, 5\}$?

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2 How many different four - digit numbers can be formed from the elements $\{2, 3, 6, 8\}$ so that the unit digit is 6?

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3 The license plates of cars in a governorate start with three letters followed by three digits except Zero. How many plates can be got assuming that there is no repetition for any letter or digit in the license plates?

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4 How many different three-digit numbers can be taken from the numbers $\{2, 5, 8, 9\}$ so that these numbers are less than 900?

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5 If you know that the set of the numbers of mobile networks in a country is made up of an eleven-digit number. If the number (025) is constant on left side, find the greatest number of phone lines which the mobile network can stand.

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6 How many ways can a different four -digit number be formed from the numbers $\{2, 3, 4, 7\}$, so that the tens digit is even.

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7 A restaurant presents 6 types of pies, 4 types of salads and 3 types of drinks. How many meals can the restaurant present daily so that a meal includes a type from each of pies, salads and drinks?

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Factorial of number n:

Factorial: The factorial of a positive integer is written as $n!$ and equals the product of all the positive integers which are less than or equal to n where:

$$n! = n(n-1)(n-2)\dots\dots\dots 3 \times 2 \times 1$$

Important Notes

➤ When $n = 0$ then $0! = 1$

➤ When $n = 1$ then $1! = 1$

➤ $n! = n(n-1)!$ where $n \in \mathbb{Z}^+$

Permutations

The number of permutations of n different objects taking r at a time is denoted by the symbol ${}^n P_r$ where:

$${}^n P_r = n(n-1)(n-2)\dots(n-r+1) \text{ where } r \leq n, r \in \mathbb{N}, n \in \mathbb{Z}^+$$

➤ ${}^n P_0 = 1$

Answer the following questions:

1 How many ways can Hossam have a meal out of three meals (Kofta - chicken - fish) and a drink out of two drinks (Juice - soft drink)? Represent it using the tree diagram.

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2 How many ways can a two -digit number be formed from the numbers 1 , 2 , 3 , 4?

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3 How many ways can a different two -digit number be formed from the numbers
1 , 2 , 3 , 4?

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4 How many ways can a different two -digit even number be formed from
the numbers 1 , 2 , 3, 4?

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5 How many ways can a committee of a man and a woman be formed
from 3 men and 4 woman?

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6 An Ice-cream shop offers three sizes (small - medium - large) and five
Flavors (strawberry, mango, lemon, milk, chocolate)
How many ways are available to buy an ice -cream cone?

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7 From the set of the letters {a , b, c , d, e , f}, find

(a) Number of ways to select a letter.

(b) Number of ways to select two letters.

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[3] Find the value of the following:

(a) $|7 \div |5$

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(c) ${}^5P_3 \times |2$

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(e) ${}^8P_1 + {}^8P_2$

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(g) ${}^5P_2 + {}^6P_3$

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(b) $3 |2 - |3$

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(d) ${}^3P_3 \times {}^2P_2$

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(f) ${}^7P_0 + {}^7P_7$

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(h) $\frac{{}^5P_5}{{}^5P_4}$

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[4] Find the value of n which satisfies the following:

(a) $|n = 24$

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(b) $\frac{|n+1}{|n-1} = 42$

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[5] Find the value of n if:

(a) ${}^7P_n = 210$

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(b) $n \underline{2n-1} = 12$

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[6] If ${}^nP_4 = 14 \times {}^{n-2}P_3$ find the value of n.

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[7] If ${}^9P_{r-1} = 504$, find the value of $\underline{r+1}$

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[8] Prove that $\frac{\underline{n+2}}{\underline{n}} = {}^{n+2}P_2$

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[9] If $\frac{1}{\underline{n}} + \frac{2}{\underline{n+1}} = \frac{56}{\underline{n+2}}$, find the value of n

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Exercises (7)

[1] Choose the correct answer:

1)	$ 0 + 1 + 2 + 3 = \dots\dots\dots$ (a) 6 (b) 8 (c) 9 (d) 10
2)	${}^9P_0 + {}^7P_1 + {}^3P_3 = \dots\dots\dots$ (a) 14 (b) 25 (c) 96 (d) 189
3)	If $\frac{1}{2} n = 1$, then $ n - 2 = \dots\dots\dots$ (a) 4 (b) 2 (c) 1 (d) zero
4)	$\frac{ n }{n} = \dots\dots\dots$ (a) 1 (b) $ n $ (c) $ n - 1 $ (d) $\frac{n}{n - 1}$
5)	nP_2 can be equal to $\dots\dots\dots$ (a) 15 (b) 16 (c) 17 (d) 20
6)	 If ${}^5P_r = 60$, then $r = \dots\dots\dots$ (a) 4 (b) 3 (c) 2 (d) 5
7)	 If ${}^nP_3 = 120$, then $n = \dots\dots\dots$ (a) 6 (b) 5 (c) 4 (d) 3
8)	 If : $ n = 1$ then $n = \dots\dots\dots$ (a) 1 (b) 0 (c) 0 or 1 (d) 1 or -1
9)	$\frac{{}^7P_r}{{}^7P_{r-1}} = \dots\dots\dots$ (a) r (b) $r - 1$ (c) $7 - r$ (d) $8 - r$

10)	If ${}^9P_{r-1} = 504$, then $\underline{r+1} = \dots\dots\dots$ (a) 5 (b) 24 (c) 120 (d) 720
11)	If $\underline{\frac{1}{2}n} = 120$, then ${}^nP_3 = \dots\dots\dots$ (a) 30 (b) 60 (c) 120 (d) 720
12)	If $\frac{1}{\underline{9}} + \frac{1}{\underline{10}} = \frac{x}{\underline{11}}$, then $x = \dots\dots\dots$ (a) 1 (b) 11 (c) 121 (d) 132
13)	The number of ordered pairs (a , b) which can be formed from the elements of the set $\{1 , 2 , 3\}$ where $a \neq b$ is $\dots\dots\dots$ (a) 2 (b) 3 (c) 6 (d) 9
14)	The number of ways to arrange 5 persons sitting in a row is $\dots\dots\dots$ (a) 4P_4 (b) 2^5 (c) 5P_4 (d) 5×5
15)	 The number of ways to arrange 4 students sitting on 4 seats in one row equals $\dots\dots\dots$ (a) 1 (b) $4 + 4$ (c) 4×4 (d) $4 \times 3 \times 2 \times 1$
16)	 If a man chooses a car from the brands $\{A , B , C\}$ and the available colors are $\{\text{white , black , silver , red}\}$, then the number of ways to choose the car equals $\dots\dots\dots$ (a) 7 (b) 12 (c) 14 (d) 24
17)	 How many numbers can be formed from three different digit from the digits 1 , 3 , 6 , 7 is $\dots\dots\dots$ (a) 9 (b) 12 (c) 24 (d) 64

18)	 How many ways can a president and a vice president be selected from a 12 members committee ? (a) 2 (b) 23 (c) 66 (d) 132
19)	 The number of ways to arrange the letters of the word «HELP» equals (a) 4 (b) 9 (c) 10 (d) 24
20)	The number of two different digit numbers can be formed from the numbers { 5 , 3 , 0 , 4 } equals (a) 3×2 (b) 4×2 (c) 3×3 (d) 3×4
21)	The number of the odd three different digit numbers formed from { 2 , 3 , 4 , 6 } equals (a) $8 \times 6 \times 3$ (b) $4 \times 3 \times 3$ (c) $4 \times 3 \times 2$ (d) $1 \times 3 \times 2$
22)	The number of ways to form a 3-digit number of 6 numbers except zero is (a) $6 \times 5 \times 4$ (b) $6 + 5 + 4$ (c) $6 \times 6 \times 6$ (d) $3 \times 2 \times 1$
23)	How many numbers can be formed from 3 different digits from the set { 2 , 4 , 5 , 7 } and be less than 500 ? (a) 7 (b) 8 (c) 12 (d) 24
24)	The number of ways can 10 true-false questions be answered is (a) <u>10</u> (b) 10 (c) 2^{10} (d) 10^2
25)	$\frac{ r-2 }{ r-3 } = \dots\dots\dots$ (a) $r - 2$ (b) $r - 3$ (c) r (d) $r - 1$

26)	If ${}^nP_n = 5040$, then $n = \dots\dots\dots$ (a) 10 (b) 5 (c) 7 (d) 3
27)	If $\lfloor n+1 \rfloor = 30 \lfloor n-1 \rfloor$, then $n = \dots\dots\dots$ (a) 5 (b) 6 (c) 29 (d) 30
28)	If $\lfloor n-1 \rfloor = 120$, $\lfloor n-r \rfloor = 6$, then ${}^nP_r = \dots\dots\dots$ (a) 720 (b) $6 \times 5 \times 4$ (c) $3 \times 4 \times 5$ (d) 6×5
29)	If $n \lfloor 2n-1 \rfloor = 60$, then $n = \dots\dots\dots$ (a) 2 (b) 2.5 (c) 4 (d) 5
30)	If $8 \times 7 \times 6 = {}^xP_y$, then $x + y$ could be equal to $\dots\dots\dots$ (a) 35 (b) 18 (c) 13 (d) 11
31)	The solution set of the equation : $\lfloor n-1 \rfloor = n$ in $\mathbb{Z}$ is $\dots\dots\dots$ (a) $\{0\}$ (b) $\{1\}$ (c) $\{0, 1\}$ (d) $\{1, 2\}$
32)	If ${}^nP_5 = {}^nP_4$, then $n = \dots\dots\dots$ (a) 20 (b) 9 (c) 4 (d) 5
33)	If $2 {}^nP_3 = {}^{n+1}P_3$, then $n = \dots\dots\dots$ (a) 4 (b) 5 (c) 6 (d) 7

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Lesson (2): Combinations

The number of combinations formed from r of objects chosen from n elements at the same time is nC_r where, $r \leq n, r \in \mathbb{N}, n \in \mathbb{Z}^+$

$$\triangleright {}^nC_r = \frac{{}^nP_r}{r!}$$

Important Corollaries

$$[1] {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$[2] {}^nC_r = {}^nC_{n-r}$$

Answer the following questions

① Calculate the value of:

(a) 6C_3

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(c) ${}^{12}C_{11}$

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(e) ${}^{12}C_9$

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(b) 9C_1

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(d) ${}^{100}C_0$

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(f) ${}^{17}C_{14}$

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② If ${}^nC_3 = 120$, find the value of ${}^nC_{n-9}$

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③ If ${}^{n+1}C_4 = \frac{5}{2} {}^nC_3$, find the value of n .

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④ If ${}^nC_3 = \frac{1}{3} 30n$, find the value of n .

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⑤ If ${}^{28}C_r = {}^{28}C_{2r-5}$, then find the value of r .

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⑥ 7 people have participated in a chess game so that a game is held between each two players. How many matches are there?

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⑦ Write in terms of permutations each of:

(a) 8C_3

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(b) ${}^{19}C_2$

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(c) 5C_0

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(d) ${}^xC_{x-y}$

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⑧ Use the form nCr to write each of:

(a) $\frac{{}^8P_2}{\underline{2}}$

(b) $\frac{{}^9P_3}{\underline{3}}$

(c) $\frac{{}^{10}P_4}{\underline{4}}$

(d) $\frac{{}^8P_0}{\underline{0}}$

Exercises (8)**[1] Choose the correct answer:**

1)	 ${}^{12}C_4 + {}^{12}C_3$ equals (a) 715 (b) 710 (c) 716 (d) 720
2)	If ${}^nC_5 : {}^nC_4 = 3 : 1$ then n equals (a) 7 (b) 9 (c) 17 (d) 19
3)	If $\underline{x-3} = 1$, then ${}^{x+2}C_{x-1} = \dots\dots\dots$ a) 3 or 4 (b) 9 or -2 (c) 10 or 20 (d) 9 or 10
4)	 If ${}^nC_3 = 120$, then ${}^nC_{n-9} = \dots\dots\dots$ (a) 90 (b) 60 (c) 10 (d) 30
5)	 If ${}^{2n-1}C_3 = 84$, then $\underline{n-5} = \dots\dots\dots$ (a) 1 (b) 2 (c) 6 (d) 24

6)	If ${}^8P_{r+1} = 336$, then ${}^{2r}C_4 = \dots\dots\dots$ (a) 16 (b) 4 (c) 2 (d) 1
7)	If ${}^nP_{n-3} = 20$, ${}^mC_n = {}^mC_{2n+1}$, then $m \times n = \dots\dots\dots$ (a) 20 (b) 40 (c) 60 (d) 80
8)	If ${}^6C_x = {}^4C_x$, then $x = \dots\dots\dots$ (a) zero (b) 1 (c) 2 (d) 3
9)	If ${}^mP_n = 24 {}^mC_n$, then $n = \dots\dots\dots$ a) 1 (b) 3 (c) 4 (d) 6
10)	If ${}^{15}C_r = {}^{15}C_6$, then $r = \dots\dots\dots$ (a) 6 (b) 9 (c) 8 (d) 6 or 9
11)	If ${}_xP_2 = {}^5C_2 \times {}^9P_2$, then $x = \dots\dots\dots$ a) 6 (b) 9 (c) 10 (d) 120
12)	If ${}^nP_r = {}^nC_r$, then $r = \dots\dots\dots$ (a) zero (b) 1 (c) zero or 1 (d) 1 or 2
13)	If ${}^nP_1 + {}^nC_2 = 36$, then $n = \dots\dots\dots$ (a) 9 (b) -9 or 8 (c) 8 (d) 9 or -8
14)	${}^nP_r : {}^nC_r = \dots\dots\dots$ (a) $\underline{n-1}$ (b) $\underline{r}$ (c) $\underline{n}$ (d) 1
15)	$\frac{{}^nC_{r-1}}{{}^nC_{r-2}} = \dots\dots\dots$ (a) $\frac{{}^nC_r}{{}^nC_{r-1}}$ (b) $\frac{n-r+1}{r}$ (c) $\frac{n-r+2}{r-1}$ (d) $\frac{{}^nP_{r-1}}{{}^nP_{r-2}}$

16)	If ${}^nC_9 : {}^nC_7 = 7 : 9$ then $n = \dots\dots\dots$ (a) 7 (b) 15 (c) 16 (d) 9
17)	 If ${}^{n-m}P_3 = 210$, ${}^{n+m}C_4 = 715$, then $m \times n = \dots\dots\dots$ (a) 15 (b) 30 (c) 35 (d) 50
18)	 If ${}^nC_{10} = {}^nC_{14}$, then ${}^{25}C_n = \dots\dots\dots$ (a) 24 (b) 25 (c) 1 (d) 49
19)	If ${}^{27}C_{2r+1} = {}^{27}C_{5r-8}$, then ${}^5C_r = \dots\dots\dots$ (a) 10 (b) 12 (c) 13 (d) 15
20)	 If ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^n$, then $n = \dots\dots\dots$ (a) 1 (b) 2 (c) 3 (d) 4
21)	$1 + {}^6C_5 + {}^7C_5 + {}^8C_5 + \dots + {}^{49}C_5 = \dots\dots\dots$ (a) ${}^{100}C_5$ (b) ${}^{99}C_6$ (c) ${}^{50}C_6$ (d) ${}^{50}C_5$
22)	If ${}^{10}C_{r+1} : {}^{10}C_{r-1} = \frac{21}{10}$, then $r = \dots\dots\dots$ (a) 3 (b) 4 (c) 5 (d) 6
23)	If ${}^{x+y}P_4 = 360$, $\underline{2x+y} = 5040$, then ${}^yC_{2x} = \dots\dots\dots$ (a) 6 (b) 8 (c) 10 (d) 12
24)	If ${}^nC_r : {}^nC_{r-1} = 7 : 4$, ${}^nP_r : {}^{n-1}P_r = 5 : 3$, then $n + r = \dots\dots\dots$ (a) 7 (b) 8 (c) 11 (d) 14
25)	If 4 friends meet each other and each shakes hands only once with each of the others , how many hands shakes will there be ? (a) 16 (b) 8 (c) 6 (d) 4

26)	 7 people have participated in a chess game so that a game is held between each two players , then the number of matches is	
	(a) 42 (b) 28 (c) 21 (d) 18	
27)	The number of diagonals in the octagon is	
	(a) 8 (b) 20 (c) 32 (d) 18	
28)	The number of sides of a polygon which has 44 diagonals is	
	(a) 7 (b) 8 (c) 11 (d) 12	
29)	 The number of ways can 7 students be selected from 10 students to take a historical trip is	
	(a) 80 (b) 120 (c) 144 (d) 70	

Unit Three

"Calculus"

3 – 1: Rate of change.

3 – 2: Differentiation.

3 – 3: Rules of differentiation.

3 – 4: Derivatives of trigonometric functions.

3 – 5: Applications on the derivative.

3 – 6: Integration.

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Lesson (1): Rate of change

The Function of variation:

If $y = f(x)$ and x varies from x to $x + h$, then the function of variation
 $v(h) = f(x + h) - f(x)$

The average rate of change function:

The average rate of change function $A(h) = \frac{f(x + h) - f(x)}{h}$

The rate of change function:

The average rate of change function $= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$

Answer the following questions

[1] If $f(x) = x^2 + 2x - 1$, find the variation in f when

(a) x varies from 2 to 2.1

(b) $x = -2$ and $h = 1$

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[2] If $f(x) = x^2 - x + 1$, find the function of variation V when $x = 3$, then calculate:

(a) $V(0.2)$

(b) $V(-0.3)$

[3] Find the average rate of change function in f when $x = x_1$, then deduce the rate of change off at the shown values of x_1 in the following:

(a) $f(x) = 2x^3$, $x_1 = 2$

(b) $f(x) = \frac{x+1}{x-1}$, $x_1 = 0$

[4] Find the average rate of change function in f where $f(x) = \sqrt{x-1}$ when $x = x_1$, Then deduce the rate of change in f when $x = 7$

[5] If $f(x) = x^2 + 3x - 1$, find:

(a) The average rate of change function when $x = 2$, then find a (0.2)

(b) The average rate of change when x varies from 4.5 to 3

[6] Find the average rate of change function in f where $f(x) = \frac{3}{x-2}$

When x varies from x_1 to $x_1 + h$, then deduce the rate of change in f when $x = 5$.

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[7] Find the average rate of change function in f where $f(x) = x^2 - 5$ when $x = x_1$,
Then deduce the rate of change in f when $x = 9$
Can the rate of change in f when $x = 5$ be calculated?

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[8] A squared lamina shrinks by cooling and maintaining its square shape.
Calculate the rate of change in the surface of lamina with respect to its side
length when the side length is 8cm.

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[9] A squared lamina expands regularly keeping its shape. Calculate
the average rate of change in its surface area when its side length varies from
3cm to 3.4cm, then calculate the rate of change in its surface area when its
side length = 5 cm.

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[10] A metal sphere expands by heating and maintaining its spherical shape. Find the rate of change in the sphere volume with respect to its radius length when the radius length is 7 cm.

[11] A sphere-shaped soap bubble expands regularly to maintain its spherical shape. Calculate the average rate of change in its spherical surface area when its radius length varies from 0.5 cm to 0.6cm, known that the surface area of the sphere equals $4\pi r^2$ where r is the sphere radius length.

[12] A triangular lamina whose base length equals twice its corresponding height. it expands by heating maintaining its shape. Find the average rate of change in its area if its height varies from 8 cm to 8.4 cm.

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Exercises (9)

[1] Choose the correct answer:

1)	If the function $f : f(x) = 3x - 2$, then the variation function $v(h) = \dots\dots\dots$ at $x = 1$ (a) 3 (b) h (c) $3h$ (d) $6h$
2)	If $f(x) = 4x + 1$, then the variation in f when x changes from 2 to 2.1 equals (a) 0.1 (b) 0.4 (c) 4 (d) 4.1
3)	If $f(x) = x^2 + 2x + 3$, then $v(h) = \dots\dots\dots$ at $x = 2$ (a) $h^2 + 2h + 3$ (b) $h^2 + 6h + 6$ (c) $2h - 1$ (d) $h^2 + 6h$
4)	 If $f(x) = x^2 - x + 1$, then : First : The variation function v when $x = 3$ is (a) $v(3)$ (b) $v(x + 3)$ (c) $2h - 1$ (d) $h^2 + 5h$ Second : $v(0.2) = \dots\dots\dots$ (a) 1.04 (b) 0.84 (c) 1.08 (d) 1.82 Third : $v(-0.3) = \dots\dots\dots$ (a) 1.41 (b) 1.31 (c) 1.04 (d) -1.41
5)	The average rate of change of the function f where $f(x) = x^2$ when x varies from 3 to 3.1 equals (a) 0.61 (b) 6.1 (c) 9 (d) 9.61
6)	 If the average rate of change in f equals 2.4 when x varies from 3 to 3.2, then the change in f equals (a) 0.32 (b) 0.48 (c) 3.6 (d) 7.2
7)	 If the average rate of change in f equals 5 when x varies from 2 to 4 and $f(2) = 6$, then $f(4) = \dots\dots\dots$ (a) -4 (b) 7 (c) 8 (d) 16

8)	If f is a function and the variation in f equals 14 when X changes from 2 to 4 , then the average rate of change in f equals	(a) 14	(b) 7	(c) $\frac{7}{2}$	(d) -7
9)	If the curve of function f passes through the two points $(1, 2)$, $(2, 3)$, then the average rate of change of the function f where X changes from 1 to 2 is	(a) 1	(b) 2	(c) zero	(d) -1
10)	If $f(2) = 5$, $f(2.3) = 7$, then the variation of f when X changes from 2 to 2.3 equals	(a) 0.3	(b) 2	(c) 35	(d) 12
11)	The radius length of a circle is r , then the average rate of change in the circumference of the circle when r changes from r_1 to r_2 is	(a) $2\pi r_1$	(b) $2\pi(r_2 - r_1)$	(c) πr_1	(d) 2π
12)	The rate of change of the function $f : f(X) = X^2 + 5X - 2$ when $X = -1$ equals	(a) 1	(b) 2	(c) 3	(d) 4
13)	The rate of change of the function $f : f(X) = \sqrt{X}$ where $X \geq 0$ at $X = 16$ equals	(a) $\frac{1}{2}$	(b) $\frac{1}{4}$	(c) $\frac{1}{8}$	(d) $\frac{1}{16}$
14)	If $f : f(X) = X^4$, then the rate of change at $X = 3$ equals	(a) 81	(b) 27	(c) 108	(d) 324
15)	 The average rate of change in the cube volume with respect to its edge length when its edge length varies from 5 cm. to 7 cm. equals	(a) 125	(b) 343	(c) 218	(d) 109

Lesson (2): Differentiation

Important definition

The rate of change is called the first derivative of the function and is denoted by the symbol $f'(x)$ where $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

The first derivative of the function is denoted by one of these symbols $\frac{dy}{dx}$ or y' or $f'(x)$

Differentiability

It is said that: the function $f'(x)$ is differentiable when $x = a$ if and only if:

$$f'(a^-) = \lim_{h \rightarrow a^-} \frac{f(a+h) - f(x)}{h} \text{ is existed, } f'(a^+) = \lim_{h \rightarrow a^+} \frac{f(+h) - f(x)}{h} \text{ is existed}$$
$$f'(a^-) = f'(a^+)$$

Remarks

- If $f(x)$ is differentiable at $x = a$, then its continuous at $x = a$
- The continuity of the function at a point does not necessarily mean that its is differentiable at this point.
- If the function is discontinuous at point $x = a$, then it is not - differentiable at $x = a$

Answer the following:

1 Using the definition of derivative, find the derivative of the function f where $f(x) = x^2 - 5$ when $x = 3$ and show the geometric meaning of the derivative of the function when $x = 3$.

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2 Find the slope of the tangent to the curve of the function f where $f(x) = x^3 - 4$ at point A $(1, -3)$, then find the measure of the positive angle which the tangent makes with the positive direction of x -axis at point A to the nearest minute.

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3 Using the definition of derivative, find the derivative of the function f where $f(x) = 1 - 5x - 3x^2$ at point $(-1, 3)$, then find the measure of the positive angle which this tangent makes with the positive direction of x -axis to the nearest minute.

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4 If $f(x) = 3x^2 + 4x + 7$, find the derivative of the function using the definition of the derivative, then find the slope of the tangent at the point $(-1, 6)$

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5 Find the derivative function for each of the following functions using the definition:

(a) $f(x) = \sqrt{3x + 1}$

(b) $f(x) = \frac{1}{x}$

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6 Prove that $f(x) = x^2 - x + 1$ is differentiable when $x = 1$

7 Discuss the differentiability of the function f where

$$f(x) = \begin{cases} 4 - x^2 & \text{when } x \leq 1 \\ 2x + 1 & \text{when } x > 1 \end{cases} \quad \text{at } x = 1$$

8 Discuss the differentiability of the function f when $x = 2$ where

$$f(x) = \begin{cases} x^2 - 5 & \text{when } x < 2 \\ 4x - 9 & \text{when } x \geq 2 \end{cases}$$

9 Discuss the differentiability of the function f where

$$f(x) = \begin{cases} x^2 + 2 & \text{when } x \geq 1 \\ 2x + 1 & \text{when } x < 1 \end{cases} \quad \text{at } x = 1$$

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10 Find the value of the constant a if the function f is differentiable at $x = 2$ where

$$f(x) = \begin{cases} 2x + 3 & \text{when } x < 2 \\ ax^2 + 8x - 1 & \text{when } x \geq 2 \end{cases}$$

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11 If the function f where $f(x) = \begin{cases} ax^2 + 1 & \text{when } x \leq 2 \\ 4x - 3 & \text{when } x > 2 \end{cases}$ is continuous at $x = 2$,

Find the value of the constant a , then discuss the differentiability of the function when $x = 2$

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12 If $f(x) = \begin{cases} x^2 + a & \text{when } x \leq 2 \\ ax + b & \text{when } x > 2 \end{cases}$ is differentiable at $x = 2$ then find $a + b$

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13 If $f(x) = ax^2 + b$ where a and b are two constants, find:

(a) The first derivative of the function f at any point (x, y) .

(b) The two values of a and b if the slope of the tangent to the curve of the function at point $(2, -3)$ lying on it equals 12.

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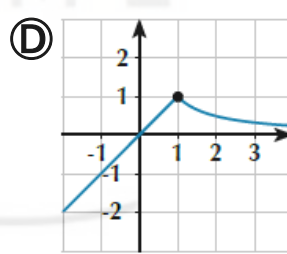
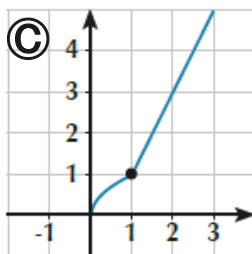
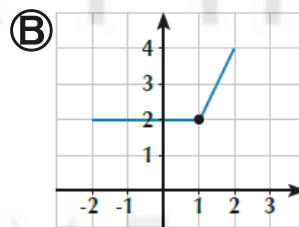
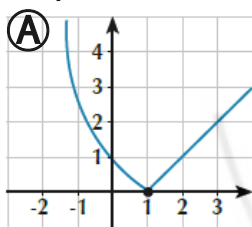
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14 Compare the right derivative and the left derivative for each of the following Function and prove that each of them is not-differentiable at point $x = 1$.



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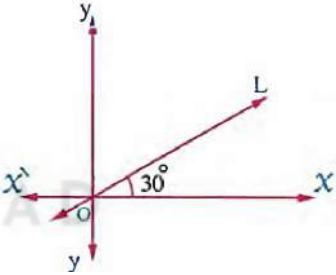
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Exercises (10)

[1] Choose the correct answer:

1)	If $\lim_{x \rightarrow 2^-} f(x) = 2k + 3$, $\lim_{x \rightarrow 2^+} f(x) = k - 1$ and the function is differentiable at $x = 2$, then $k = \dots\dots\dots$ (a) 2 (b) -2 (c) 4 (d) -4
2)	The slope of the tangent to the curve of the function f at the point (x_1, y_1) which lies on it equals $\dots\dots\dots$ (a) $f(x_1 + h) - f(x_1)$ (b) $\frac{f(x_1 + h) - f(x_1)}{h}$ (c) $\lim_{h \rightarrow 0} \frac{f(x_1 + h) - f(x_1)}{h}$ (d) $\lim_{x \rightarrow 0} \frac{f(x_1) - f(h)}{x_1 - h}$
3)	All the following statements are false except $\dots\dots\dots$ (a) If the function is continuous at a point, then it is differentiable at this point. (b) If the function is not differentiable at a point, then the function is undefined at this point. (c) If the function f is not continuous at a point, then the function is not differentiable at this point. (d) If the function has right derivative and left derivative at a point, then it is differentiable at this point.
4)	If f is a function and $f(1) = 5$, $\tilde{f}(1) = 4$, then $\lim_{x \rightarrow 1} f(x) = \dots\dots\dots$ (a) 5 (b) 4 (c) 9 (d) not exist.
5)	The function $f : f(x) = x - 5$ at $x = 5$ $\dots\dots\dots$ (1) is continuous. (2) is differentiable. (3) has a limit. (a) (1) only. (b) (1), (2) only. (c) (1), (3) only. (d) (1), (2) and (3)
6)	If $f(x) = \begin{cases} x^2 + 3 & , \quad x \geq 1 \\ x^2 - 3 & , \quad x < 1 \end{cases}$, then $\tilde{f}(1^+) = \dots\dots\dots$ (a) 2 (b) -2 (c) 4 (d) not exist.

7)	If $f(x) = \begin{cases} x^2 & , \quad x \leq 2 \\ 4x & , \quad x > 2 \end{cases}$, then $f'(2) = \dots\dots\dots$ (a) 2 (b) 4 (c) 8 (d) not exist.	
8)	 If $f : f(x) = \begin{cases} x^2 + a & , \quad x \leq 2 \\ ax + b & , \quad x > 2 \end{cases}$ is differentiable at $x = 2$, then $a + b = \dots\dots\dots$ (a) 4 (b) -4 (c) -8 (d) 8	
9)	 If $f(x) = \begin{cases} x^2 + x + 1 & , \quad x \geq 1 \\ 3x & , \quad x < 1 \end{cases}$, then the function f at $x = 1$ (a) is continuous but not differentiable. (b) is differentiable. (c) has a vertical tangent. (d) has a horizontal tangent.	
10)	If the function f is continuous at $x = 3$ and $\lim_{x \rightarrow 3} \frac{2x - 6}{f(x) - f(3)} = \frac{1}{6}$, then $f'(3) = \dots\dots\dots$ (a) -12 (b) 10 (c) 12 (d) -10	
11)	If f is a function where $f(2 + h) = f(2) + h$, then $f'(2) = \dots\dots\dots$ (a) zero (b) 1 (c) 2 (d) undefined.	
12)	If the curve of the function f in the given figure is represented by the straight line L, then the average rate of change of the function f is (a) $\tan 30^\circ$ (b) $\sin 30^\circ$ (c) $\cos 30^\circ$ (d) $-\tan 30^\circ$ 	

Lesson (3): Rules of differentiation

The derivatives of some functions

- $f(x) = a \rightarrow f'(x) = 0$
- $f(x) = x^n \rightarrow f'(x) = n x^{n-1}$
- $f(x) = ax^n \rightarrow f'(x) = an x^{n-1}$

[1] Complete:

- Ⓐ $\frac{d}{dx} \left(\frac{1}{x^3} \right) = \dots\dots\dots$
- Ⓑ $\frac{d}{dx} (5\pi) = \dots\dots\dots$
- Ⓒ $\frac{d}{dx} (x^2) = \dots\dots\dots$
- Ⓓ $\frac{d}{dx} \left(\frac{1}{\sqrt[3]{x}} \right) = \dots\dots\dots$
- Ⓔ $\frac{d}{dx} (5x^2 + 3x + 2) = \dots\dots\dots$
- Ⓕ $\frac{d}{dx} (\sqrt{2}x^7 - \frac{x^5}{5} + \pi) = \dots\dots\dots$
- Ⓖ $\frac{d}{dx} (-\sqrt{2}) = \dots\dots\dots$
- Ⓗ $\frac{d}{dx} \left(\frac{4}{3} \pi x^3 \right) = \dots\dots\dots$
- Ⓘ $\frac{d}{dx} (\sqrt[3]{x^5}) = \dots\dots\dots$
- Ⓙ $\frac{d}{dx} \left(\frac{-4}{x^5} \right) = \dots\dots\dots$

[2] Find the first derivative for each of the following functions with respect to x.

Ⓐ $y = 2x^6 + 3\sqrt{x}$

Ⓑ $y = \frac{4x^2 - 3x + 2\sqrt{x}}{x}$

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© $y = x(3x^2 - \sqrt{x})$

© $y = \frac{4x^2 - x + 3}{\sqrt{x}}$

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© $y = (x - 1)(x + 1)(x^2 + 1)(x^4 + 1)(x^8 + 1)(x^{16} + 1)$

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© $y = 3x^8 - 2x^5 + 6x + 1$

© $y = \frac{5}{x} + x\sqrt{x} + \sqrt{3}x - 4$

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WITH SHERIF SAAD

Derivative of the product of two functions

$$\text{If } y = g(x) \cdot z(x) \quad \rightarrow \quad \frac{dy}{dx} = g \cdot \frac{dz}{dx} + z \cdot \frac{dg}{dx}$$

Derivative of the quotient of two functions

$$\text{If } y = \frac{g(x)}{z(x)} \quad \rightarrow \quad \frac{dy}{dx} = \frac{z \times \frac{dg}{dx} - g \times \frac{dz}{dx}}{[z(x)]^2}$$

[3] Find the first derivative for each of the following functions

Ⓐ $y = (x^2 + 3)(x^3 - 3x + 1)$

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Ⓑ $y = (x^2 - \sqrt{x})(x^2 + 2\sqrt{x})$

Ⓒ $y = (2x^4 - 3x + 4)(x^2 - \sqrt{x} + \frac{2}{x})$

Ⓓ $y = \frac{5x - 2}{5x + 1}$

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$$\textcircled{E} \quad y = \frac{x^2 + 2x + 5}{x^2 - 5x + 1}$$

$$\textcircled{F} \quad y = \frac{x - 2}{x + 5}$$

$$\textcircled{G} \quad y = (4x^2 - 1)(7x^3 + x) \text{ when } x = 1$$

$$\textcircled{H} \quad y = \frac{x^3 + 2x^2 - 1}{x + 5}$$

Derivative of the composite function

$$\text{If } y = [f(x)]^n \quad \rightarrow \quad \frac{dy}{dx} = n[f(x)]^{n-1} f'(x)$$

Chain rule

$$\text{If } y = f(z) , z = f(x) \quad \rightarrow \quad \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

[4] Find the value of $\frac{dy}{dx}$ at the point shown in each of the following:

Ⓐ $y = \left(\frac{x^2 - 2}{3 + x^2}\right)^7$ when $x = 0$

Ⓑ $y = (x^2 + 1)^5 (x^2 - x + 1)^{-4}$ when $x = 1$

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Ⓒ $y = \left(\frac{5x^2}{3x^2 + 2}\right)^3$

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WITH SHERIF SAAD

[5] If $y = ax^3 + bx^2$ and $\frac{dy}{dx} = 8$ when $x = 1$ and the average rate of change of y when x varies from -1 to 2 equals 7 , find the values of the two constant a and b

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[6] Find the value of $\frac{dy}{dx}$ when $x = 2$ for each of the following:

Ⓐ $y = (x^3 + x - 9)^5$

Ⓑ $y = 3x^2 - 4$

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Ⓒ $y = \sqrt[3]{(2x^3 + 4x + 3)^2}$

Ⓓ $y = z^2$, $z = 3x^2 - 2$

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$$\textcircled{E} \quad y = 2z^3, \quad z = 8x - 11$$

$$\textcircled{F} \quad y = \frac{z-1}{z+1}, \quad z = \frac{x+1}{x-1}$$

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Exercises (11)

[1] Choose the correct answer:

1)	If $f(x) = \frac{1}{x}$, then $f'(1) = \dots\dots\dots$ (a) 1 (b) zero (c) -1 (d) 2
2)	$\frac{d}{dx} [(x-1)(x+3)] = \dots\dots\dots$ (a) $2x$ (b) $2x+1$ (c) $2x-3$ (d) $2x+2$
3)	If $f(x) = \frac{1}{2}x^{14} - \frac{1}{4}x^{20} + \frac{1}{6}x^{36}$, then $f'(1) = \dots\dots\dots$ (a) 6 (b) 8 (c) 16 (d) 18
4)	If $f(x) = (4x^2 - 1)(1 + 4x^2)$, then $f'\left(-\frac{1}{2}\right) = \dots\dots\dots$ (a) zero (b) -1 (c) -8 (d) -64
5)	If $f'(2) = 5$, $f(x) = x^2 + m x + 5$, then $m = \dots\dots\dots$ (a) 5 (b) 2 (c) 10 (d) 1
6)	If $f(x) = x^5$, then $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \dots\dots\dots$ (a) 5 (b) $5x^4$ (c) x^5 (d) not exist.

7)	If $y = \frac{1}{\sqrt{x}}$, then $\frac{dy}{dx} = \dots\dots\dots$ (a) $\frac{-1}{2\sqrt{x^2}}$ (b) $\frac{-1}{2x}$ (c) $\frac{-1}{2x\sqrt{x}}$ (d) $\frac{1}{2\sqrt{x}}$	
8)	If $f(x) = a^2$ where a is a constant, then $f'(x) = \dots\dots\dots$ (a) zero (b) $2a$ (c) 2 (d) a^2	
9)	If $y = \frac{x^2+6}{x^3+5}$, then $\dot{y} = \dots\dots\dots$ at $x = 1$ (a) $\frac{-1}{4}$ (b) $\frac{6}{5}$ (c) $\frac{7}{6}$ (d) $\frac{11}{12}$	
10)	If $y = (x+4)(x^2+5x+7)$, then $\dot{y} = \dots\dots\dots$ (a) $2x^2+13x+20$ (b) $x^2+18x+20$ (c) $x^3+18x^2+27x+28$ (d) $3x^2+18x+27$	
11)	If $f(x) = x + \frac{9}{x}$, then $\dot{f}(x) = \text{zero at } x = \dots\dots\dots$ (a) 3 (b) -3 (c) ± 3 (d) ± 9	
12)	If $f(x) = 4x^2 + g(x)$ and $\dot{f}(3) = 7$, then $\dot{g}(3) = \dots\dots\dots$ (a) 24 (b) 31 (c) -17 (d) -24	
13)	If $f(x) = \frac{1}{3}x^3 - 2x^2 + 5x - 4$ and $\dot{f}(x) = 2$, then $x = \dots\dots\dots$ (a) 1 or 2 (b) 1 or 3 (c) 2 or 3 (d) 3 or 4	
14)	If $f(x) = 2ax^3$, $\dot{f}(-3) = 108$, then $\dot{f}(1) = \dots\dots\dots$ (a) 2 (b) 6 (c) 12 (d) 24	

15)	If $y = \frac{x^2 + 2x + 4}{x^2 - 2x + 4}$ and $\frac{dy}{dx} = \text{zero}$, then $x = \dots\dots\dots$ (a) ± 2 (b) ± 4 (c) 2 or 3 (d) -2 or -3
16)	If $f(x) = x^3 - ax + 1$, $g(x) = ax^2 + bx + 3$ and $2f'(1) + g'(1) = 2$, then $b = \dots\dots\dots$ (a) -4 (b) -3 (c) 3 (d) 4
17)	If $\frac{d}{dx} [(ax)^3] = 24$ at $x = 1$, then the value of a equals $\dots\dots\dots$ (a) 8 (b) 2 (c) 1 (d) 24
18)	If $f(x) = x^3 - 3x^2 - 5x + 6$, then $\lim_{h \rightarrow 0} \frac{f(1+2h) - f(1)}{h} = \dots\dots\dots$ (a) -8 (b) 8 (c) 16 (d) -16
19)	If $f(x) = 3x^2 - 5x + 1$ and $f(a) = f'(a)$, then $a = \dots\dots\dots$ (a) 3 or $\frac{2}{3}$ (b) $\frac{2}{3}$ or $\frac{1}{3}$ (c) 1 or 3 (d) 2 or 3
20)	If $y = (ax + 2)^2$ and $\frac{dy}{dx} = 6$ at $x = 1$, then $a = \dots\dots\dots$ (a) 1 or 6 (b) 1 or -3 (c) -3 or 2 (d) 2 or 6
21)	If $y = x^2$, $x = z^3 - 1$, then $\frac{dy}{dz} = \dots\dots\dots$ (a) $z^3 - 1$ (b) $6z^2$ (c) $6z^2(z^3 - 1)$ (d) $x^2(z^3 - 1)$
22)	If $y = x^2 + x$, $x = z^2 + z$, then $\frac{dy}{dz} = \dots\dots\dots$ at $z = -1$ (a) -1 (b) 1 (c) zero (d) 2

23)	If $y = z^3 - z - 1$, $z = x - \frac{4}{x}$, then $\frac{dy}{dx} = \dots\dots\dots$ at $x = 2$ (a) -1 (b) -2 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$
24)	If $z = \frac{1}{2} y^2 + y - 2$, $y = x^2 + x + 1$, then $\frac{dz}{dx} = \dots\dots\dots$ at $x = -\frac{1}{2}$ (a) zero (b) 1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$
25)	If $y = \sqrt{z} + \frac{1}{\sqrt{z}}$, $z = x^2 + 25$, then $\frac{dy}{dx} = \dots\dots\dots$ at $x = 0$ (a) -2 (b) zero (c) 1 (d) 3
26)	If $y = \frac{z+3}{z-1}$, $z = \frac{x+1}{x-3}$, then $\frac{dy}{dx} = \dots\dots\dots$ at $x = 4$ (a) zero (b) 1 (c) 2 (d) 3
27)	 If $f(x) = \sqrt{x^2 + 9}$, then $f'(-4) = \dots\dots\dots$ (a) $-\frac{4}{5}$ (b) 5 (c) $\frac{1}{10}$ (d) $-\frac{1}{10}$
28)	If $y = (z+1)^3$, $z = x^5 - 1$, then $\frac{dy}{dx} = \dots\dots\dots$ (a) x^{15} (b) x^8 (c) $15x^{14}$ (d) $8x^7$
29)	 If $y = (z-1)^5$, $z = x^2 + 3$, then $\frac{dy}{dx} = \dots\dots\dots$ (a) $10x(x^2+2)^4$ (b) $5(x^2+2)^4$ (c) $5(2x+2)^4$ (d) $10x(x^2+2)^5$

Lesson (4): Derivatives of trigonometric functions

The derivatives of some functions

(1) If $y = \sin z$, then $\frac{dy}{dz} = \cos z \cdot \frac{dz}{dx}$

(4) If $y = \cot z$, then $\frac{dy}{dz} = (-\csc^2 z) \cdot \frac{dz}{dx}$

(2) If $y = \cos z$, then $\frac{dy}{dz} = -\sin z \cdot \frac{dz}{dx}$

(5) If $y = \sec z$, then $\frac{dy}{dz} = (\sec z \tan z) \cdot \frac{dz}{dx}$

(3) If $y = \tan z$, then $\frac{dy}{dz} = \sec^2 z \cdot \frac{dz}{dx}$

(6) If $y = \csc z$, then $\frac{dy}{dz} = (-\csc z \cot z) \cdot \frac{dz}{dx}$

Find the first derivative of each of the following:

(1) $y = \sin 3x$

(2) $y = 2 \cos (4 - x)$

(3) $y = \cot (2x - 1)$

(4) $y = \sec (3x + 5)$

(5) $y = \sin (2x + \pi) + \tan 3x$

(6) $y = x^4 + \tan (5x - 2)$

(7) $y = \sin (3x^3 + 2x + 1)$

(8) $y = \cos \frac{x}{3} + \cos \frac{\pi}{3}$

(9) $y = 5x^2 + \csc \sqrt{x}$

Find the first derivative of each of the following:

(1) $y = 5 \cot (3 - \pi x)$

(3) $y = 3 \tan (2x + 3)$

(5) $y = 2 \cos (4 - 3x^2)$

(7) $y = \sin \frac{\pi}{4} - 7 \sin x$

(9) $y = \tan (\sqrt{x})$

(11) $y = 3x^5 + 4 \cot x$

(2) $y = \cos (5x + 3)$

(4) $y = \tan (-5x^2 + 19)$

(6) $y = x - \sin 2x + \tan 5x$

(8) $y = \cos (2x - 1) + \sin (4 - 5x)$

(10) $y = \sin \left(\frac{1}{x^2} \right)$

(12) $y = 3 \sec 2x + 2 \csc 3x$

Find the first derivative of each of the following:

(1)  $y = x \sin(3x - 2)$

(3)  $y = x \sin x - 3 \cos x$

(5) $y = (3x + 4) \csc x$

(7) $y = x \sin x + \sqrt{x}$

(9) $y = 2 \sin(2x + 1) \cos(2x + 1)$

(11)  $y = \frac{\tan x}{x}$

(13) $y = \frac{1 + \cos 2x}{1 - \cos 2x}$

(15) $y = \frac{\sec x}{1 + \sec x}$

(2) $y = x^2 \sin 2x$

(4) $y = x \sec 3x$

(6)  $y = x \tan(3x) + \cos x$

(8) $y = 2 \sin 2x \cos 3x$

(10)  $y = x^2 \sin(2x^2 + 5)$

(12) $y = \frac{\sin 2x}{\cos x}$

(14) $y = \frac{\cot x}{x}$

Exercises (12)

[1] Choose the correct answer:

1)	 If $f(x) = \tan(5x - \pi)$, then $\dot{f}\left(\frac{\pi}{4}\right) = \dots\dots\dots$ (a) 5 (b) $5\sqrt{2}$ (c) 10 (d) $10\sqrt{3}$
2)	If $y = \csc 2x$, then $\frac{dy}{dx} = \dots\dots\dots$ at $x = \frac{\pi}{6}$ (a) $-\frac{3}{4}$ (b) $-\frac{4}{3}$ (c) $-\frac{1}{2}$ (d) $\sqrt{3}$
3)	 $\frac{d}{dx}(\cos^2 x + \sin^2 x) = \dots\dots\dots$ (a) 1 (b) zero (c) x (d) $2(\cos x + \sin x)$
4)	 $\frac{d}{dx}\left(\tan \frac{\pi}{4}\right) = \dots\dots\dots$ (a) 1 (b) $\sec^2 \frac{\pi}{4}$ (c) zero (d) 2
5)	If $f(x) = \cot(5x - \pi)$, then $\dot{f}\left(\frac{\pi}{4}\right) = \dots\dots\dots$ (a) $5\sqrt{2}$ (b) $-5\sqrt{2}$ (c) 10 (d) -10
6)	If $y = \sec\left(\frac{\pi}{4}x\right)$, then $\frac{dy}{dx} = \dots\dots\dots$ (a) $\frac{\pi}{4} \sec\left(\frac{\pi}{4}x\right) \tan\left(\frac{\pi}{4}x\right)$ (b) $-\frac{\pi}{4} \sec\left(\frac{\pi}{4}x\right) \tan\left(\frac{\pi}{4}x\right)$ (c) $\frac{\pi}{4} \sec\left(\frac{\pi}{4}\right) \tan\left(\frac{\pi}{4}\right)$ (d) $\sec\left(\frac{\pi}{4}x\right) \tan\left(\frac{\pi}{4}x\right)$
7)	 If $y = 2 \sin(3x + 4)$, then $\dot{y} = \dots\dots\dots$ (a) $6 \cos(3x + 4)$ (b) $2 \cos(3x + 4)$ (c) $6 \cos(3x)$ (d) $-6 \cos(3x + 4)$

8)	 If $y = \sin (X^2 + 3)$, then $\frac{d y}{d X} = \dots\dots\dots$ (a) $2 \cos (X^2 + 3)$ (b) $2 \cos X$ (c) $-2 \cos X$ (d) $2 X \cos (X^2 + 3)$
9)	 If $y = (\sin X + \cos X)^2$, then $\frac{d y}{d X} = \dots\dots\dots$ (a) $2 \sin 2 X$ (b) $2 \cos 2 X$ (c) $2 \sin X$ (d) $\cos X - \sin X$
10)	 If $y = \sin^2 X - \cos^2 X$, then $\frac{d y}{d X} = \dots\dots\dots$ (a) $2 \sin 2 X$ (b) $2 \cos 2 X$ (c) $2 \sin X - 2 \cos X$ (d) $\cos X - \sin X$
11)	 If $y = \tan^2 3 X$, then $\dot{y} = \dots\dots\dots$ (a) $2 \tan 3 X$ (b) $6 \tan 3 X$ (c) $6 \tan 3 X \sec^2 3 X$ (d) $3 \sec^4 3 X$
12)	If $y = 16 \sin \frac{X}{4} \cos \frac{X}{4} \cos \frac{X}{2} \cos X$, then $\frac{d y}{d X} = \dots\dots\dots$ (a) $\frac{1}{2} \sin 2 X$ (b) $\cos 2 X$ (c) $2 \sin 2 X$ (d) $4 \cos 2 X$
13)	If $y = \frac{\sin^2 X}{1 + \cos X}$, then $\frac{d y}{d X} = \dots\dots\dots$ (a) $1 - \cos X$ (b) $\sin^2 X$ (c) $\sin X$ (d) $1 + \cos X$
14)	If $y = \sin (\cos^2 X)$, then $\dot{y} = \dots\dots\dots$ (a) $\cos (\cos^2 X)$ (b) $\cos (\cos^2 X) \times 2 \cos X$ (c) $\cos^3 X \times -2 \cos X$ (d) $-2 \sin X \cos X \cos (\cos^2 X)$

Lesson (5): Applications on the derivative

Geometric applications

- If $y = f(x)$, then the slope of the tangent to the curve of the function f at any point on it $= \frac{dy}{dx}$
- If (x_1, y_1) is a point on the curve of the function $y = f(x)$, then the equation of the tangent at point (x_1, y_1) is given by the relation $y - y_1 = m(x - x_1)$.

where $m = \frac{dy}{dx}$ is the slope of the tangent at point (x_1, y_1) .

- The equation of the normal on the tangent at point (x_1, y_1) is given by the relation $y - y_1 = \frac{-1}{m}(x - x_1)$. where $m = \frac{dy}{dx}$ is the slope of the tangent at point (x_1, y_1) .

Answer the following questions

- 1** Find the measure of the positive angle, which the tangent to the curve of $y = x^2 + \frac{1}{x} - 1$ makes with the positive direction of x-axis when $x = 1$

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2 Find the measure of the positive angle, which the tangent to the curve of $y = \frac{x+3}{x-2}$ makes with the positive direction of x-axis at point (3, 6)

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3 Find the measurement of the positive angle which the normal on the curve of $y = \sqrt{2x^2 + 7}$ makes with the positive direction of x-axis at point (-3, 5) to the nearest minute.

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TIME

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6 Find the points which lie on the curve of $y = x^2 - 2x + 3$ at which the tangent to the curve is:

(a) Parallel to x -axis

(b) Perpendicular to the straight-line $x - 4y + 1 = 0$

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7 Find the equation of the tangent to the curve of the function $y = (x - 2)(x + 1)$ at its two intersection points with x-axis.

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8 Find the two equations of the tangent and normal to the curve of

$y = \frac{x+3}{x+1}$ at the point lying on the curve and whose abscissa = 1.

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9 Find the equation of the normal on the tangent to the curve

$y = \frac{x^2-1}{2-x^2}$ when $x = 0$

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10 Find the equations of the tangent and normal to the curve of

$$y = x \sin^2 x \text{ at point } \left(\frac{\pi}{4}, \frac{\pi}{8}\right)$$

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11 Find the equation of the tangent to the curve of $y = 2 \sin x + \cos x$ at point $(0, 1)$.

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12 Prove that the tangent drawn to the curve of $y = x^2 + x - 1$ at point $(1, 1)$ is perpendicular to the tangent drawn to the curve of $y = 2 - \sqrt[3]{x}$ at the same point.

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13 If the curve of $y = (x^2 - 2x)(a x + b)$ touches the x -axis at point $(2, 0)$, and touches the straight-line $y = 2x$ at the origin point, find the two values of a and b .

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14 Find the value of the two constants a and b if the slope of the tangent to the curve of $y = x^2 + a x + b$ at point $(1, 3)$ lying on it equals 5

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Exercises (13)

[1] Choose the correct answer:

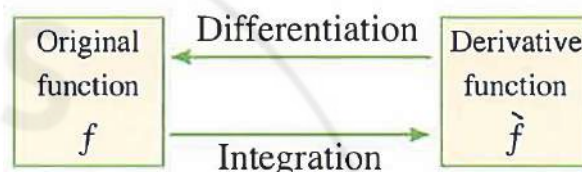
1)	 The slope of the tangent to the curve of the function $y = (2x - 3)^5$ at $x = 2$ equals (a) 1 (b) $\frac{1}{12}$ (c) 5 (d) 10	
2)	 The slope of the tangent to the curve of the function $y = \sin x \cos x$ equals (a) $\cos x \sin x$ (b) $\cos^2 x - \sin^2 x$ (c) $\sin x - \cos x$ (d) $\sin^2 x + \cos^2 x$	
3)	The slope of the normal to the curve $y = \sin 2x$ at the point with x -coordinate $\frac{\pi}{6}$ and lies on the curve equals (a) zero (b) -1 (c) 3 (d) 1	
4)	 The point on the curve of the function $f : f(x) = (x - 3)^2 - 1$ at which the tangent is parallel to the straight line : $2x + y - 3 = 0$ is (a) $(0, 2)$ (b) $(2, 0)$ (c) $(3, -1)$ (d) $(1, 3)$	
5)	If the tangent to the curve of the function $y = f(x)$ at the point $(3, 5)$ passes through the point $(6, -1)$, then $f'(3) = \dots\dots\dots$ (a) -2 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 2	
6)	If the tangent to the curve of the function f where $f(x) = ax^2 + bx + 5$ at the point $(-1, 3)$, which lies on it, makes with the positive direction of the x -axis a positive angle of measure 45° , then $a + b = \dots\dots\dots$ (a) 1 (b) 2 (c) 3 (d) 4	
7)	If the slope of the tangent to the curve : $y = ax^2 + 2bx$ at the origin equals 6 and the point $(-1, -3)$ lies on the curve, then $a + b = \dots\dots\dots$ (a) 3 (b) 4 (c) 6 (d) 8	

8)	Slope of normal to the curve $y = (x - 1)(x^3 + 2)$ at $x = 1$ equals	(a) $-\frac{1}{3}$	(b) -3	(c) $\frac{1}{3}$	(d) 3
9)	 Measure of the positive angle that the tangent to the curve $y = \sqrt{2x^2 + 7}$ at the point $(-3, 5)$ makes with the positive direction of x -axis $\approx$	(a) $39^\circ 48'$	(b) $50^\circ 12'$	(c) $129^\circ 48'$	(d) $140^\circ 12'$
10)	 Equation of the normal to the curve $y = \tan x$ at the point $(\frac{\pi}{4}, 1)$ which lies on it is	(a) $y - 2x = 1 - \frac{\pi}{2}$	(b) $8y + 4x = \pi + 8$	(c) $4y - 4x = \pi + 4$	(d) $4y + 2x = \pi + 4$
11)	The points on the curve $y^2 = 8x$ at which $\frac{dy}{dx} = \frac{dx}{dy}$ are	(a) $(2, 4), (4, -2)$	(b) $(2, 4), (2, -4)$	(c) $(-2, 4), (2, 4)$	(d) $(4, 2), (4, -2)$
12)	The points on the curve $y^2 = 8x$ at which $\frac{dy}{dx} = \frac{dx}{dy}$ are	(a) $(2, 4), (4, -2)$	(b) $(2, 4), (2, -4)$	(c) $(-2, 4), (2, 4)$	(d) $(4, 2), (4, -2)$
13)	 If the straight line $y = 8 - 3x$ is a tangent to the curve of the function f at the point $(3, -1)$, then $\ddot{f}(3) =$	(a) -1	(b) -3	(c) 3	(d) 8

Lesson (6): Integration

Anti-derivative of the function

- It is said that: the function f is an anti-derivative to the function f' if $f'(x) = f'(x)$ for each x in the domain of f .
- If $f'(x) = f'(x)$, then $\int f'(x) dx = f(x) + C$ where C is an arbitrary constant.



Properties of integration

If f and g are integrable functions on an interval, then:

- 1- $\int a f(x) dx = a \int f(x) dx$ where a is a constant $\neq 0$
- 2- $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$

Standard formula for integration

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Important Rules for integration

$$[1] \int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C \quad \text{where} \quad n \neq -1$$

Calculate the following integrations:

(1) $\int 9x^8 dx$

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(2) $\int 12x^{-2} dx$

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$$(3) \int 5 \sqrt{x^3} \, dx$$

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$$(5) \int x^8 \, dx$$

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$$(7) \int \sqrt[7]{x^5} \, dx$$

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$$(9) \int 2(3z^2 - 5) \, dz$$

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$$(11) \int (3\sqrt{x} - \frac{2}{x^3}) \, dx$$

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$$(13) \int x(x + 3) \, dx$$

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$$(4) \int -\frac{8}{5} c^{-3} \, dc$$

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$$(6) \int x^{\frac{2}{3}} \, dx$$

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$$(8) \int 7x^{\frac{7}{9}} \, dx$$

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$$(10) \int (3x^2 + x - 2) \, dx$$

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$$(12) \int (ax^2 + 5x + c) \, dx$$

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$$(14) \int x^2 (3x - \frac{5}{x}) \, dx$$

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$$(15) \int (2 + \sqrt{x} + \frac{1}{\sqrt{x}}) dx$$

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$$(17) \int (x - 2)(x + 2) dx$$

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$$(19) \int (\sqrt{x} - 1)^2 dx$$

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$$(21) \int \frac{2x^2 + 3}{x^2} dx$$

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$$(23) \int \frac{x^2 - 1}{x - 1} dx$$

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$$(25) \int \frac{x^3 + 8}{x^2 - 2x + 4} dx$$

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$$(16) \int (\frac{1}{x^2} + \sqrt{x} + 3) dx$$

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$$(18) \int (x - 5)(x + 1) dx$$

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$$(20) \int \left(x + \frac{1}{x}\right)^2 dx$$

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$$(22) \int \frac{x^7 - 4x^5 + x^3}{x^3} dx$$

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$$(24) \int \frac{x^3 - 27}{x - 3} dx$$

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$$(26) \int \frac{x^2 - 5x + 6}{x - 2} dx$$

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$$(27) \int 6(x-2)^5 dx$$

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$$(29) \int (8-3x)^4 dx$$

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$$(31) \int \frac{12}{(2x-5)^4} dx$$

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$$(33) \int (x-1)(x^2-2x+1) dx$$

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$$(35) \int (3x+5)^{-9} dx$$

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$$(37) \int (x^2+3x-2)^9 (2x+3) dx$$

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$$(28) \int 3(x+3)^{-4} dx$$

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$$(30) \int (2z-3)^{\frac{7}{3}} dz$$

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$$(32) \int \frac{7}{\sqrt{x+4}} dx$$

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$$(34) \int \sqrt{x-3} (x-3)^4 dx$$

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$$(36) \int (x+1)(x+3)^7 dx$$

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$$(38) \int x(4x^3-3x^2+4)^5 (2x-1) dx$$

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integration of some trigonometric functions

$$(1) \int \cos X \, dX = \sin X + C$$

$$(3) \int \sec^2 X \, dX = \tan X + C$$

$$(5) \int \sec X \cdot \tan X \, dX = \sec X + C$$

$$(2) \int \sin X \, dX = -\cos X + C$$

$$(4) \int \csc^2 X \, dX = -\cot X + C$$

$$(6) \int \csc X \cdot \cot X \, dX = -\csc X + C$$

Important corollaries

$$(1) \int \sin (aX + b) \, dX = -\frac{1}{a} \cos (aX + b) + C$$

$$(2) \int \cos (aX + b) \, dX = \frac{1}{a} \sin (aX + b) + C$$

$$(3) \int \sec^2 (aX + b) \, dX = \frac{1}{a} \tan (aX + b) + C$$

$$(4) \int \csc^2 (aX + b) \, dX = -\frac{1}{a} \cot (aX + b) + C$$

$$(5) \int \sec (aX + b) \tan (aX + b) \, dX = \frac{1}{a} \sec (aX + b) + C$$

$$(6) \int \csc (aX + b) \cot (aX + b) \, dX = -\frac{1}{a} \csc (aX + b) + C$$

Find:

$$(1) \int \cos (5X + 6) \, dX$$

$$(3) \int \left(5 \cos X - 7 \sin \frac{X}{2} \right) \, dX$$

$$(5) \int \left(3X^2 + \frac{1}{\cos^2 X} + \sin \frac{\pi}{6} \right) \, dX$$

$$(7) \int \sec X (\sec X + \tan X) \, dX$$

$$(2) \int \sin \left(\frac{X}{2} + 3 \right) \, dX$$

$$(4) \int \left(3 - X + \sec^2 \frac{1}{4} X \right) \, dX$$

$$(6) \int (1 + \sec X) (1 - \sec X) \, dX$$

$$(8) \int \csc X (\cot X + \csc X) \, dX$$

Find:

(1) $\int \left(\sec^2 \frac{x}{4} + \sec^2 \frac{\pi}{4} \right) dx$

(3) $\int (5 + \tan^2 x) dx$

(5) $\int \sin^2 x dx$

(7) $\int \left(\cos x \cos \frac{\pi}{3} + \sin x \sin \frac{\pi}{3} \right) dx$

(2) $\int [1 + \cot^2 (2x - 5)] dx$

(4) $\int (\sin x + \cos x)^2 dx$

(6) $\int (1 + \cos x)^2 dx$

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Exercises (14)**[1] Choose the correct answer:**

1)	$\int x \, dx = \dots\dots\dots + C$ (a) x^2 (b) $\frac{1}{2} x$ (c) $\frac{1}{2} x^2$ (d) 1
2)	$\int y^8 \, dy = \dots\dots\dots + C$ (a) $8 y^7$ (b) $\frac{1}{9} y^9$ (c) zero (d) y^9
3)	$\int (x+2) \, dx = \dots\dots\dots + C$ (a) $2 x^2 + 2 x$ (b) $x^2 + 2 x$ (c) $\frac{1}{2} x^2 + 2 x$ (d) $(x+2)^2$
4)	$\int (-3) \, dx = \dots\dots\dots + C$ (a) $-3 x$ (b) x (c) $\frac{(-3)^2}{2}$ (d) $\frac{-3}{2} x$
5)	The antiderivative of the function $f : f(x) = 3x^2 - 2x + 5$ is (a) $6x - 2$ (b) $3x^2 - 2x + 3$ (c) $x^3 - x^2 + 5x + C$ (d) $\frac{1}{3} x^3 - \frac{1}{2} x^2 + 5x + C$
6)	$\int x^{\frac{2}{3}} \, dx = \dots\dots\dots + C$ (a) $\frac{3}{5} x^{\frac{5}{3}}$ (b) $\frac{5}{3} x^{\frac{5}{3}}$ (c) $\frac{2}{3} x^{-\frac{1}{3}}$ (d) $x^{\frac{5}{3}}$
7)	 $\int \sqrt[7]{x^5} \, dx = \dots\dots\dots + C$ (a) $\frac{12}{7} x^{\frac{12}{7}}$ (b) $\sqrt[7]{\frac{x^6}{6}}$ (c) $\frac{1}{6} x^6$ (d) $\frac{7}{12} x^{\frac{12}{7}}$

8)	 $\int 5\sqrt{x^3} \, dx = \dots\dots\dots + C$ (a) $2\sqrt{x^5}$ (b) $\frac{5}{2}\sqrt{x^4}$ (c) $5x^{\frac{5}{2}}$ (d) $\frac{5}{2}x^2$
9)	 $\int (x^2 - 3x - 1) \, dx = \dots\dots\dots + C$ (a) $\frac{1}{3}x^3 + 3x - x$ (b) $\frac{1}{3}x^3 - \frac{3}{2}x^2 - x$ (c) $x^3 - 3x^2 - x$ (d) $\frac{1}{2}(x^2 - 3x - 1)^2$
10)	 $\int 2(3z^2 - 5) \, dz = \dots\dots\dots + C$ (a) $(3z^2 - 5)^2$ (b) $6z^2 - 10$ (c) $2z^3 - 10z$ (d) $z^2 - 5z$
11)	 $\int (x - 5)(x + 1) \, dx = \dots\dots\dots + C$ (a) $\frac{1}{3}x^3 - 2x^2 - 5x$ (b) $x^3 - 2x^2 - 5x$ (c) $\frac{2}{3}x^3 - 2x^2 + 5x$ (d) $\frac{1}{3}x^3 - 2x^2 + 5x$
12)	 $\int (x + 2)(x - 2) \, dx = \dots\dots\dots$ (a) $x + 4 + C$ (b) $\frac{1}{3}x^3 - 4x + C$ (c) $x^3 - 4x + C$ (d) $(x^2 - 4)^2 + C$
13)	 $\int (x^2 - 2)^2 \, dx = \dots\dots\dots + C$ (a) $\frac{1}{3}(x^2 - 2)^3$ (b) $2(x^2 - 2)(2x)$ (c) $\frac{1}{5}x^5 - \frac{4}{3}x^3 + 4x$ (d) $x^4 - 4x^2 + 4$
14)	$\int \cos(4x) \, dx = \dots\dots\dots + C$ (a) $\frac{1}{4}\sin(4x)$ (b) $\frac{1}{2}\cos(2x)$ (c) $\frac{1}{4}\cos^2(4x)$ (d) $4\sin(4x)$

15)	$\int \sin (3 x) d x = \dots\dots\dots + C$ (a) $\frac{1}{3} \sec ^2 (3 x)$ (c) $\frac{1}{3} \cos (3 x)$	(b) $-\frac{1}{3} \cos (3 x)$ (d) $\frac{1}{3} \sin (3 x)$
16)	$\int \sec ^2 (5 x) d x = \dots\dots\dots + C$ (a) $\frac{1}{15} \sec ^3 (5 x)$ (c) $\frac{1}{5} \tan (5 x)$	(b) $\frac{1}{2} \tan ^2 (2 x)$ (d) $\frac{1}{5} \cot (5 x)$
17)	$\int \csc ^2 (3 x) d x = \dots\dots\dots + C$ (a) $-\cot (3 x)$ (c) $-3 \cot (3 x)$	(b) $\frac{1}{3} \cot (3 x)$ (d) $\frac{-1}{3} \cot (3 x)$
18)	$\int \sec y \tan y d y = \dots\dots\dots + C$ (a) $\sec y$ (c) $\tan y$	(b) $\csc y$ (d) $\cot y$
19)	 $\int \sin (3 x-5) d x = \dots\dots\dots + C$ (a) $-\cos (3 x-5)$ (c) $\frac{-1}{3} \cos (3 x-5)$	(b) $-3 \cos (3 x-5)$ (d) $\frac{-2}{3} \cos (3 x-5)$
20)	 $\int (x-\sin x) d x = \dots\dots\dots + C$ (a) $3 \sin \left(\frac{x}{3}-2\right)$ (c) $\frac{1}{2} x^2-\cos x$	(b) $\frac{1}{2} x^2+\cos x$ (d) $1+\cos x$

21)	 $\int \sin \left(\frac{x}{4} + \frac{\pi}{3} \right) dx = \dots\dots\dots + C$ (a) $-4 \cos \left(\frac{x}{4} + \frac{\pi}{3} \right)$ (c) $\frac{1}{2} \cos \left(\frac{x}{4} + \frac{\pi}{3} \right)$	(b) $\cos \left(\frac{x}{4} + \frac{\pi}{3} \right)$ (d) $-4 \cos \left(\frac{x}{2} + \frac{\pi}{3} \right)$
22)	 $\int (\cos^2 x - \sin^2 x) dx = \dots\dots\dots + C$ (a) $\frac{1}{3} \cos^3 x - \frac{1}{3} \sin^3 x$ (c) $\frac{1}{2} \sin 2x$	(b) $\cos x$ (d) $2 \sin x$
23)	 $\int (\sin x + \cos x)^2 dx = \dots\dots\dots + C$ (a) $x - \frac{1}{2} \cos 2x$ (c) $x + \sin^2 x$	(b) $x + \frac{1}{2} \cos^2 x - \frac{1}{2} \sin^2 x$ (d) $x - \cos^2 x$

Unit Four

"Trigonometry"

(4 - 1): Angles of elevation and depression

(Application on solving the triangle).

(4 - 2): Trigonometric functions of sum and difference of the measures of two angles.

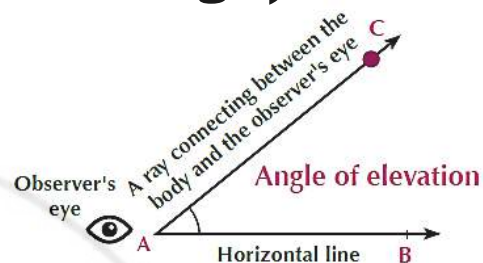
(4 - 3): The trigonometric functions of the double-angle.

Lesson (1): Angles of elevation and depression

(Application on solving the triangle)

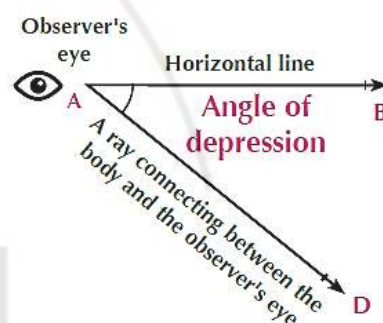
Angle of elevation

When you see an object above you, there's an **angle of elevation** between the horizontal and your line of sight to the object



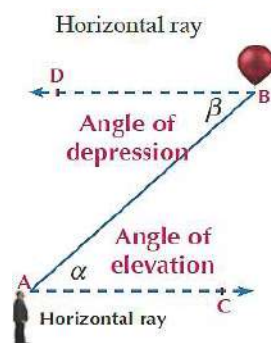
Angle of depression

When you see an object below you, there's an **angle of depression** between the horizontal and your line of sight to the object



In the opposite figure

Angle of elevation = Angle of depression



Remember that

The sine Law

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The sine law used in the following cases

- Two angles and a side
- Two sides and non-included angle

The cosine Law

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = c^2 + a^2 - 2ca \cos B \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$c^2 = a^2 + b^2 - 2ab \cos C \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

The cosine law used in the following cases

Three sides

Two sides and an included angle

Answer the following questions

1 A man observed the angle of elevation of a tower top from a point on the ground surface to find its measurement is $20^{\circ}35'$. He walked on a horizontal road in the direction of the tower base for a distance of 50m to find the measurement of the angle of elevation of the tower top is 42° . Find the height of the tower to the nearest meter.

2 From a top of a house of height 15 m, the measurement of the angle of elevation of a tower top is 67° and the measurement of the angle of depression of a tower base is 35° . Find the height of the tower to the nearest meter known that the tower base and the house base are at the same horizontal level.

3 From a point on the ground surface a man observed the top of a tower at an angle of elevation of 20° , He walked on a horizontal way in the direction of the tower base for 50 meters, the measurement of the angle of elevation of the tower top is 42° . Find the height of the tower to the nearest meter

4 A man observed the angle of elevation of a tower peak from a point on the ground surface to find its measurement is 22° , then he walked on a horizontal road towards the tower base for a distance of 50 meters to observe the angle of elevation of the tower peak once more to find its measurement is 36° . Find the height of the tower to the nearest meter.

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5 From a tower top of height 65 meters, the two angles of depression of points A and B at the horizontal level are measured to give 32° and $21^\circ 12'$ respectively. If D represents the tower base and $A \in \overline{BD}$, find the length of $\overline{AB}$ to the nearest meter.

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6 From a hill top, an observer found that the measurement of the two angles of depression of a tower top and its base are 15° and 26° respectively. If the height of the tower is 50 meters, calculate the height of the hill to the nearest meter known that the two bases of the hill and tower are at the same horizontal level.

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7 From a hill peak, an observer found the measurement of the two angles of depression of a tower peak and its base are $15^\circ 18'$ and $26^\circ 42'$ respectively.

If the height of the tower is 20 meters, find the height of the hill to the nearest meter .

8 A ship on the sea observed a lighthouse to find that it is located at a distance of 50 km towards the East, then the ship sailed in the direction of North of East and after two hours the lighthouse lied in the direction of 50° South of East.

Calculate the velocity of the ship.

9 A tower of height 12 meters is constructed on a hill. If the measures of the two angles of elevation of the top and base of the tower from a point on the ground surface are 32° and 24° respectively, find the height of the hill to the nearest meter.

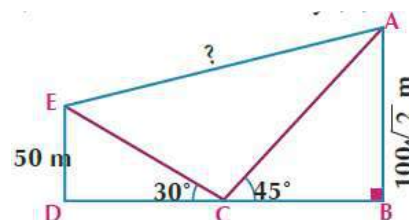
10 A lighthouse of height 60 meters is constructed on a hill close to the seashore. M The two angles of elevation of the top and base of the lighthouse were measured from a boat on the sea to give 70° and 45° respectively. Find the height of the hill from the sea level to the nearest meter.

11 From tower peak of 10 meters high, a man measured the two angles of Elevation and depression of the highest and the lowest points of a minaret on the Ground surface to find them $14^\circ 28'$ and 42° respectively. If the height of the tower and minaret were founded at the same horizontal level, calculate the height of the minaret to the nearest meters.

12 In the opposite figure:

two balloons A and E of height $100\sqrt{2}$ and 50 meters.

A body (C) on the ground lying at the vertical level passing through the balloons is observed. If the measurement of the two angles of depression of the body are 45° and 30° respectively, find the distance between the two balloons to the nearest meter.



13 From the top of a rock 80 meter height, the two angles of depression of the top and base of a tower were measured to give 24° and 35° respectively.

Find the height of the tower known that the bases of the rock and tower are at the same horizontal level

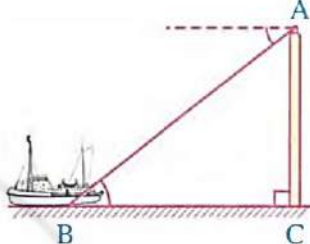
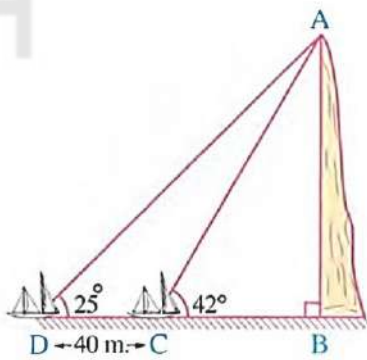
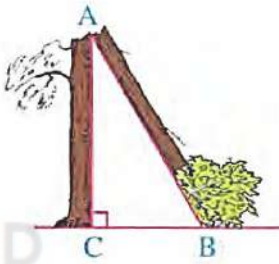
14 From the top a rock of height 80 meters, the two angles of depression of the top and the base of a tower were measured to give 24° and 35° respectively.

Find the height of the tower to the nearest meters known that the two bases of the rock and tower are in the same horizontal level.

15 From a top of a house, A man observed a car at the rest at the same horizontal level of the house to find the measure of the angle of depression is 70° , when this man descends vertically down for a distance of 12 meters, he observed the measure of the angle of depression of the car was 30° . Find the height of the house to the nearest meter and the distance between the car and house.

Exercises (15)

[1] Choose the correct answer:

1)	From the top of a lighthouse , a boat was observed at an angle of depression of measure 38° , if the horizontal distance between the boat and the base of the lighthouse is 220 m. , then the height of the lighthouse $\simeq$ m. (a) 164 (b) 172 (c) 186 (d) 196	
2)	From a point on the ground surface a man observed the elevation angle of the top of a building to be of measure θ , then the measure of the depression angle of the man position from the top of the building is (a) θ (b) $90^\circ - \theta$ (c) $\theta - 90^\circ$ (d) $180^\circ - \theta$	
3)	A man in a boat , observed the angle of elevation of the top of a rock of measure 25°. He moved horizontally towards the base of the rock 40 m. He then measured the elevation angle of the top of the rock and it was of measure 42° , then the height of the rock $\simeq$ m. (a) 39 (b) 42 (c) 46 (d) 51	
4)	Because of the wind , the upper part of a tree is broken and makes an angle of measure 60° with the ground , if the point of contact of the top of the tree with the ground was at a distance 10 m. from its bottom , then the tree height $\simeq$ m. (a) 32 (b) 35 (c) 37 (d) 42	

5) From the midpoint between two bases of a building and a tree of height 15 m. , a man found that the measures of the two angles of elevation of the top of the tree and the top of the building were 30° , 60° respectively , then the height of the building = m.

- (a) 40 (b) $15\sqrt{3}$ (c) $30\sqrt{3}$ (d) 45

6) From an aeroplane , it was found that the measure of the angle of depression of a fixed point on the ground was 60° When the aeroplane came down at a distance 200 m. , it was found that the measure of the angle of depression of the same point became 45° , then the height of the aeroplane at the same moment of the first measurement $\simeq$ m.

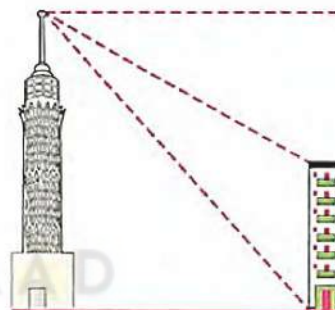
- (a) 453 (b) 473 (c) 493 (d) 512

7)  (Marine Navigation) A ship sailed from a certain point in the direction of 12° South of the East with velocity 11 km./h. At the same moment another ship sailed from the same point in the direction of 68° North of the East with velocity 6.5 km./h. , then the distance between the two ships after 2 hours from the time of their sailing together $\simeq$ km.

- (a) 24 (b) 26 (c) 28 (d) 30

8) From the top of cairo tower , a man used the scope to observe the angles of depression of the top and the base of his house , which is located near to the tower , were 45° , 60° respectively. If the height of the tower is 187 m. , then the height of the house $\simeq$ metre.

- (a) 96 (b) 87 (c) 108 (d) 79



Lesson (2): Trigonometric functions of sum and difference of the measures of two angles

Sin & cosine Functions

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$

tan Function

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Related angles

$$\sin (180 - A) = \sin A$$

$$\cos (180 - A) = -\cos A$$

$$\sin (180 + A) = -\sin A$$

$$\cos (180 + A) = -\cos A$$

Related angles

$$\sin (-A) = -\sin A$$

$$\cos (-A) = \cos A$$

$$\tan (-A) = -\tan A$$

Basic trigonometric Functions

$$\sin x = \frac{1}{\csc x}$$

$$\cos x = \frac{1}{\sec x}$$

$$\tan x = \frac{1}{\cot x} \text{ \& } \tan x = \frac{\sin x}{\cos x}$$

Basic trigonometric identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

Answer the following questions:

[1] Evaluate without using calculator:

Ⓐ $\cos 105^\circ$

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Ⓑ $\sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ$

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Ⓒ $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$

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Ⓓ $\sin \frac{2\pi}{3} \cos \frac{\pi}{6} - \cos \frac{2\pi}{3} \sin \frac{\pi}{6}$

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Ⓔ $\frac{\sin 25^\circ \cos 20^\circ + \cos 25^\circ \sin 20^\circ}{\sin 80^\circ \cos 35^\circ + \cos 80^\circ \sin 35^\circ}$

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Exercises (16)

[1] Choose the correct answer:

1)	$\cos A \cos B + \sin A \sin B = \dots\dots\dots$ (a) $\sin (A + B)$ (b) $\sin (A - B)$ (c) $\cos (A + B)$ (d) $\cos (A - B)$
2)	$\sin A \cos B - \cos A \sin B = \dots\dots\dots$ (a) $\sin (A + B)$ (b) $\cos (A + B)$ (c) $\sin (A - B)$ (d) $\cos (A - B)$
3)	$\sin (A + B) + \sin (A - B) = \dots\dots\dots$ (a) $2 \sin A$ (b) $2 \sin^2 A$ (c) $2 \cos A \sin A$ (d) $2 \sin A \cos B$
4)	$\sin \frac{\pi}{9} \cos \frac{\pi}{18} + \cos \frac{\pi}{9} \sin \frac{\pi}{18} = \dots\dots\dots$ (a) $\frac{1}{2}$ (b) 2 (c) $\frac{\sqrt{2}}{2}$ (d) $\frac{\sqrt{3}}{2}$
5)	If $\frac{\tan X - \tan Y}{1 + \tan X \tan Y} = 2$, then $\tan (X - Y) = \dots\dots\dots$ (a) $\frac{1}{2}$ (b) 2 (c) $\sqrt{2}$ (d) $\frac{\sqrt{3}}{3}$
6)	$\frac{\tan 50^\circ + \tan 10^\circ}{1 - \tan 50^\circ \tan 10^\circ} = \dots\dots\dots$ (a) $\tan 40^\circ$ (b) $\tan 60^\circ$ (c) $\tan 50^\circ \times \tan 10^\circ$ (d) $2 \tan 50^\circ \tan 10^\circ$
7)	$\frac{1 + \tan A}{1 - \tan A} = \dots\dots\dots$ (a) $\tan \left(\frac{\pi}{4} + A \right)$ (b) $\tan \left(\frac{3\pi}{4} + A \right)$ (c) $\tan \left(\frac{\pi}{2} + A \right)$ (d) $\tan \left(\frac{5\pi}{4} - A \right)$

8)	 $\sin \left(\theta + \frac{\pi}{6} \right) = \dots\dots\dots$ (a) $\frac{1}{2} (\cos \theta + \sqrt{3} \sin \theta)$ (b) $\frac{1}{2} (\cos \theta + \sin \theta)$ (c) $\frac{1}{2} (\sqrt{3} \cos \theta + \sin \theta)$ (d) $\frac{1}{2} (\sqrt{3} \cos \theta + \sqrt{2} \sin \theta)$
9)	 $\sin 5 X \sin 3 X + \cos 5 X \cos 3 X = \dots\dots\dots$ (a) $\cos 2 X$ (b) $\cos 8 X$ (c) $\sin 8 X$ (d) $\sin 2 X$
10)	If $\tan \theta = \frac{1}{2}$, then $\tan (\theta + 45^\circ) = \dots\dots\dots$ (a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) 3 (d) 6
11)	 If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, then $\tan (A + B) = \dots\dots\dots$ (a) 1 (b) $-\frac{5}{6}$ (c) $\frac{1}{6}$ (d) $\frac{5}{6}$
12)	 $\cot \left(2\pi - \frac{5\pi}{12} \right) = \dots\dots\dots$ (a) $2 + \sqrt{3}$ (b) $2 - \sqrt{3}$ (c) $-2 - \sqrt{3}$ (d) $\sqrt{3} - 2$
13)	$\cos X \cos \left(X + \frac{\pi}{3} \right) + \sin X \sin \left(X + \frac{\pi}{3} \right) = \dots\dots\dots$ (a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $\frac{-\sqrt{3}}{2}$ (d) $\frac{-1}{2}$
14)	If $\tan A + \tan B = 4 - 4 \tan A \tan B$, then $\tan (A + B) = \dots\dots\dots$ (a) $\frac{1}{2}$ (b) 4 (c) 1 (d) 2
15)	$\frac{\tan (A - B) + \tan B}{1 - \tan (A - B) \tan B} = \dots\dots\dots$ (a) $\tan A$ (b) $-\tan (A + B)$ (c) $\tan (A + B)$ (d) $-\tan B$

16)	In ΔABC , $\cos A = \frac{-3}{5}$, $\sin B = \frac{5}{13}$, then $\sin C = \dots\dots\dots$	
	(a) $\frac{32}{65}$ (b) $\frac{17}{65}$ (c) $\frac{33}{65}$ (d) $\frac{3}{13}$	
17)	If $\sec A = \frac{5}{4}$, $\csc B = \frac{13}{5}$, where A and B are the measures of two acute angles , then $\sec (A - B) = \dots\dots\dots$	
	(a) $\frac{56}{65}$ (b) $\frac{63}{65}$ (c) $\frac{-33}{65}$ (d) $\frac{65}{63}$	
18)	If $\tan^2 A = \frac{9}{16}$, where $A \in \left[0, \frac{\pi}{2}\right[$, $\tan^2 B = \frac{1}{9}$ where $B \in \left[\frac{\pi}{2}, \pi\right[$, then $\tan (A + B) = \dots\dots\dots$	
	(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{9}$ (d) $\frac{2}{9}$	

MATH

TIME

WITH SHERIF SAAD

Lesson (3): The trigonometric functions of the double-angle

Sin & Cosine Functions

$$\sin 2A = 2\sin A \cos A$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

tan Function

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Answer the following questions:

[1] Find without using the calculator the values of: $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$ if:

Ⓐ $\sin \theta = \frac{4}{5}$, $0^\circ < \theta < 90^\circ$

Ⓑ $\cos \theta = \frac{1}{3}$, $0^\circ < \theta < \frac{\pi}{2}$

Ⓒ $\csc \theta = \frac{5}{3}$, $\pi < \theta < \frac{3\pi}{2}$

Ⓓ $\cot \theta = \frac{3}{2}$, $180^\circ < \theta < 270^\circ$

[2] Find without using the calculator the values of : $\sin 2\theta$, $\cos 2\theta$,

$\sin \frac{\theta}{2}$ $\cos \frac{\theta}{2}$ if:

Ⓐ $\cos \theta = \frac{1}{4}$, $0^\circ < \theta < 90^\circ$

Ⓑ $\sin \theta = \frac{3}{5}$, $90^\circ < \theta < 180^\circ$

Ⓒ $\tan \theta = \frac{4}{3}$, $\pi < \theta < \frac{3\pi}{2}$

Ⓓ $\sin \theta = \frac{15}{17}$, $\frac{3\pi}{2} < \theta < 2\pi$

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[3] If $\cos A = \frac{4}{5}$, $0^\circ < A < 90^\circ$, find the values for each of the following
 $\sin 2A$, $\cos 2A$, $\tan 2A$

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[4] If you know that $\cos A = \frac{5}{13}$, where A is a measurement of a positive acute angle, **find** $\sin 2A$, $\cos 2A$, $\tan 2A$ $\cos \frac{A}{2}$

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[5] If $\sin A = \frac{-12}{13}$, where $\frac{3\pi}{2} < A < 2\pi$, without using the calculator find the value of $\sin 2A$, $\cos 2A$

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WITH SHERIF SAAD

Exercises (17)

[1] Choose the correct answer:

1)	$\cos^2 X - \sin^2 X = \dots\dots\dots$ (a) 1 (b) $\sin 2X$ (c) $\cos 2X$ (d) $2 \sin X \cos X$
2)	$\sin 2A = \dots\dots\dots$ (a) $2 \sin A$ (b) $2 \sin A \cos A$ (c) $1 - 2 \sin^2 A$ (d) $\sin^2 A$
3)	$\tan 2A = \dots\dots\dots$ (a) $2 \tan A$ (b) $2 \tan A \cot A$ (c) $\frac{2 \tan A}{1 - \tan^2 A}$ (d) $\frac{2 \tan A}{1 + \tan^2 A}$
4)	$\sin 6X \cos 6X = \dots\dots\dots$ (a) $\sin 12X$ (b) $\frac{1}{2} \sin 12X$ (c) zero (d) $\sin 3X$
5)	If $\frac{\tan X}{1 - \tan^2 X} = \frac{5}{2}$, then $\tan 2X = \dots\dots\dots$ (a) $\frac{5}{2}$ (b) 5 (c) $\frac{2}{5}$ (d) 2
6)	$(\tan 2A)(\cot A)(1 - \tan^2 A) = \dots\dots\dots$ (a) 1 (b) $\sin A$ (c) $\tan A$ (d) 2
7)	 $1 + \cos 4A = \dots\dots\dots$ (a) $2 \cos^2 4A$ (b) $\cos^2 2A$ (c) $\cos^2 4A$ (d) $2 \cos^2 2A$
8)	 $1 - 2 \sin^2 50^\circ = \dots\dots\dots$ (a) $\sin 100^\circ$ (b) $\cos 50^\circ$ (c) $\cos 100^\circ$ (d) $\sin 50^\circ$
9)	 If $\cos C = \frac{1}{3}$, then $\cos 2C = \dots\dots\dots$ (a) zero (b) $-\frac{2}{3}$ (c) $-\frac{7}{9}$ (d) $\frac{2}{3}$

10)	 $\cos^2 C - \cos 2 C = \dots\dots\dots$ (a) $\sin C$ (b) $\cos C$ (c) $\sin^2 C$ (d) $\tan C$
11)	$2 \sin (X + 45^\circ) \cos (X + 45^\circ) = \dots\dots\dots$ (a) $2 \sin (X + 45^\circ)$ (b) $\cos (90^\circ + 2 X)$ (c) $\sin 2 X$ (d) $\cos 2 X$
12)	If $\sin X + \cos X = 1$, then $\sin 2 X = \dots\dots\dots$ (a) 3 (b) 2 (c) 1 (d) zero
13)	$\csc 2 X = \dots\dots\dots$ (a) $\sec X \csc X$ (b) $\frac{1}{2} \sec X \csc X$ (c) $\sin X \cos X$ (d) $2 \sin X \cos X$
14)	$\csc^2 X (1 - \cos 2 X) = \dots\dots\dots$ (a) 2 (b) 1 (c) $\frac{1}{2}$ (d) - 2
15)	$\frac{1}{2} (\tan \theta + \cot \theta) = \dots\dots\dots$ (a) $\cos 2 \theta$ (b) $\sin 2 \theta$ (c) $\csc 2 \theta$ (d) $\sec 2 \theta$
16)	$\frac{1 - \tan^2 X}{1 + \tan^2 X} = \dots\dots\dots$ (a) $\cos 2 X$ (b) $\sin 2 X$ (c) $\cos X$ (d) $\sin X$

WITH SHERIF SAAD

Application Math

2nd Secondary

2nd term

Unit One

"Rectilinear Motion"

lesson 1: Rectilinear motion.

lesson 2: Uniformly accelerated rectilinear motion.

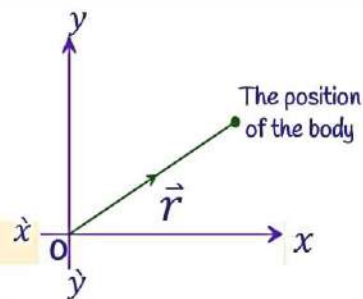
lesson 3: Vertical motion under the effect of gravity (Free fall).

Lesson 1: Rectilinear motion

The position vector of a particle

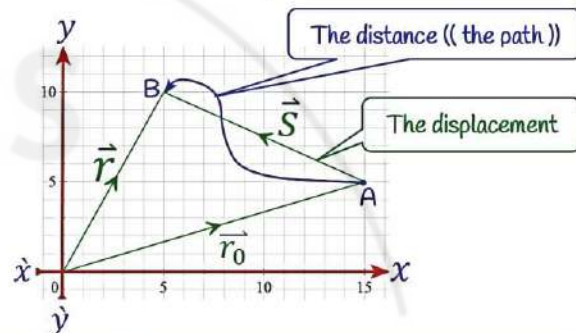
The position vector of a body is the vector which its initial point coincides the position of the observer (O) and its end point coincides the position of the body simultaneously

The position vector of a body is denoted by ($\vec{r}$) where : $\vec{r} = x\vec{i} + y\vec{j}$



The displacement and the distance:

If a car moved from the initial position (A) to reach the final point (the end point) (B) following the path shown on the opposite figure then :



The displacement	The distance
It is the vector represented by the directed line segment AB where its starting point (A) coincides with the initial position of the body and its ending point (B) on its end position, the displacement AB is denoted by S	(The covered distance) is the length of the path which the body moves. The covered distance is a scalar quantity.

The relation between the position vector and the displacement:

From the graph: $\therefore \vec{r}_0 + \vec{S} = \vec{r} \quad \therefore \vec{S} = \vec{r} - \vec{r}_0$

For example:

A body moved such that its position vector $\vec{r}$ is given as a function in (t) in terms of the two basic unit vectors $\vec{i}$ and $\vec{j}$ by the relation $\vec{r} = (t + 2)\vec{i} + (3t - 2)\vec{j}$ Find :

- (1) The displacement S
- (2) The norm of the displacement till the moment $t = 4$ seconds.
- (3) The norm of the displacement between the two moments $t = 2$ and $t = 4$

Solution:

$$(1) \therefore \vec{S} = \vec{r} - \vec{r}_0 \quad \therefore \vec{r}_0 = (0 + 2)\vec{i} + (3(0) - 2)\vec{j} = (2)\vec{i} + (-2)\vec{j}$$

$$\therefore \vec{S} = \vec{r} - \vec{r}_0 = [(t + 2)\vec{i} + (3t - 2)\vec{j}] - [(2)\vec{i} + (-2)\vec{j}] = (t)\vec{i} + (3t)\vec{j}$$

$$(2) \therefore \vec{S}_4 = (4)\vec{i} + (12)\vec{j} \quad \therefore \|\vec{S}_4\| = \sqrt{4^2 + 12^2} = 4\sqrt{10}$$

$$(3) \therefore \vec{S}_4 - \vec{S}_2 = (4 - 2)\vec{i} + (12 - 6)\vec{j} = 2\vec{i} + 6\vec{j} \quad \therefore \|\vec{S}\| = \sqrt{2^2 + 6^2} = 2\sqrt{10}$$

The velocity - The speed

The velocity	The speed
The velocity is a vector quantity expresses the rate of change in the position of the body with respect to time.	Speed is the scalar quantity which expresses the norm of the velocity.

The units of measuring the speed:

They are : $\text{km./hr.} \xrightarrow{\times \frac{5}{18}} \text{m./sec.} \xrightarrow{\times 100} \text{cm./sec.}$ then $\text{km./hr.} \xrightarrow{\times \frac{250}{9}} \text{cm./sec.}$
 $\text{cm./sec.} \xrightarrow{\div 100} \text{m./sec.} \xrightarrow{\times \frac{18}{5}} \text{km./hr.}$ then $\text{cm./sec.} \xrightarrow{\times \frac{9}{250}} \text{km./hr.}$

for example:

$$27 \text{ km./hr.} = 27 \times \frac{5}{18} \text{ m./sec.} = 7.5 \text{ m./sec.}$$

$$4.5 \text{ km./hr.} = 4.5 \times \frac{250}{9} \text{ cm./sec.} = 125 \text{ cm./sec.}$$

$$20 \text{ m./sec.} = 20 \times \frac{18}{5} \text{ km./hr.} = 72 \text{ km./hr.}$$

The uniform motion: ((1) Constant direction , (2) Constant magnitude)

In the case of uniform motion we find that :

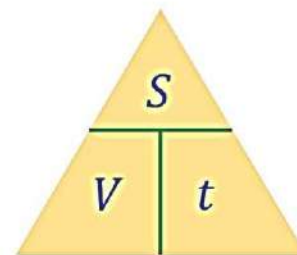
(1) The norm of the displacement = the covered distance

(2) The relation between the displacement ($\vec{s}$) and the velocity ($\vec{v}$) is

$$\vec{s} = t \vec{v} \quad \text{i.e.} \quad s = t v \quad \text{or} \quad v = \frac{s}{t} \quad \text{or} \quad t = \frac{s}{v}$$

(3) The constant velocity in this case is called the uniform velocity :

which it means that : the body covers equal displacements in equal times.



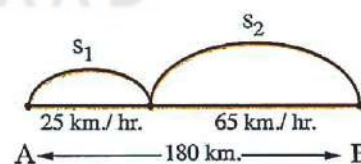
Two cities A and B a straight road between them. A car moved from A towards B with speed 25 km./hr. at the same moment another car moved from B towards A with speed 65 km. / hr. Find when and where do the two cars meet. Knowing that the length of the road is 180 km.

solution:

Assuming that the two cars meet after t hours.

$$\therefore s_1 + s_2 = 180 \quad \therefore 25t + 65t = 180$$

$$\therefore 90t = 180 \quad \therefore t = \frac{180}{90} = 2 \text{ hr.}$$



The two cars meet after 2 hours from the moment of starting motion.

$$\therefore s_1 = 25 \times 2 = 50 \text{ km.} \quad \text{The two cars meet at a distance 50 km. far from A}$$

The average velocity -The average speed

The average velocity	The average speed
The velocity is a vector quantity expresses the rate of change in the position of the body with respect to time. $\vec{V}_A = \frac{\text{The happend displacement}}{\text{The total time}} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$	The average Speed (V_A) is the scalar quantity i.e. $V_A = \frac{\text{The total distance}}{\text{The total time}}$

Notice that

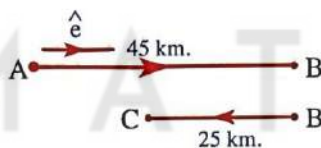
It is not necessary that the average speed equals the norm of the average velocity.

For Example: A car covered a distance 45 km. on a straight road in $\frac{3}{4}$ hour then it returned back to cover 25 km. in the opposite direction in $\frac{1}{2}$ hour Find at the end of the tour :

- (1) The displacement happened. (2) The total covered distance.
 (3) The average velocity. (4) The average speed.

Solution:

Assume the unit vector $\hat{e}$ in the direction from A to B then.



(1) The displacement happened $= (45)\hat{e} - (25)\hat{e} = (20)\hat{e}$	(2) The total covered distance = $45 + 25 = 70$ km.
(3) The average velocity $\vec{V}_A = \frac{\text{The happend displacement}}{\text{The total time}} = \frac{20}{0.75+0.5} = \frac{20}{1.25} = 16 \hat{e}$	(4) The average speed $V_A = \frac{\text{The total distance}}{\text{The total time}} = \frac{70}{0.75+0.5} = 56 \text{ km.}$

i.e. The **average velocity** has the same direction of $\hat{e}$ and its norm = 16 km./hr.

For Example: In an orthogonal coordinate system, a particle begins its motion from a fixed point. After 3 seconds from the beginning of motion the particle was at the position A (7 , 3) and after 5 seconds from the beginning of motion the particle was at the position B (13 , 11). Find the average velocity of the particle during this interval then find its magnitude and its direction.

Solution:

$$\vec{r}_1 = \vec{OA} = (7, 3) \quad \text{and} \quad \vec{r}_2 = \vec{OB} = (13, 11)$$

$$\therefore \vec{V}_A = \frac{\text{The happend displacement}}{\text{The total time}} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{(13, 11) - (7, 3)}{5 - 3} = \frac{(6, 8)}{2} = (3, 4)$$

$$\therefore \|\vec{V}_A\| = \sqrt{3^2 + 4^2} = 5 \text{ lengthunit/sec.}$$

$$\therefore \tan(\theta) = \frac{4}{3} \quad \therefore \theta = \tan^{-1}\left(\frac{4}{3}\right) = 53^\circ \hat{7} 48''$$

i.e. The direction of the average velocity makes an angle of measure $53^\circ \hat{7} 48''$ with the positive direction of x -axis

The relative speed: (The concept of relative velocity)

The relative velocity for a particle (B) with respect to another particle (A) is the velocity that the particle (B) appears to move with if we assume that the particle (A) is at rest and it is denoted by (V_{BA})

i.e. $\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$

i.e. The relative velocity of B with respect to A = the velocity of B – the velocity of A

note that $\vec{V}_B = \vec{V}_{BA} + \vec{V}_A$

For Example:

① A car moves on a straight road with velocity 75 km./hr. a bicycle moves on the same road with velocity 45 km./hr. Find the velocity of the bicycle relative to the car.

If: (1) The bicycle moves in opposite direction of the car.

(2) The bicycle moves in the same direction of the car.

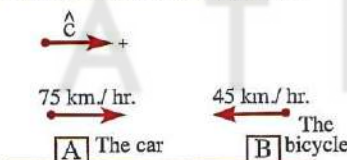
Solution:

Assume the unit vector $\hat{c}$ in the direction of the car.

(1) The bicycle B moves in the opposite direction of the car A.

$\therefore \vec{V}_A = 75 \hat{c} \quad , \quad \vec{V}_B = -45 \hat{c}$

$\therefore \vec{V}_{BA} = \vec{V}_B - \vec{V}_A$
 $= (-45 - 75)\hat{c} = -120 \hat{c}$

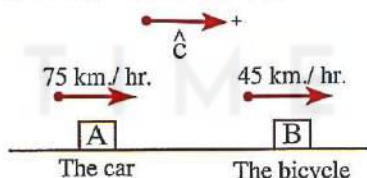


i.e. The bicycle seems to the driver of the car as it moves towards him as it is in the opposite direction of the car with velocity of magnitude 120 km./hr.

(2) The bicycle B moves in the same direction of the car A.

$\therefore \vec{V}_A = 75 \hat{c} \quad , \quad \vec{V}_B = 45 \hat{c}$

$\therefore \vec{V}_{BA} = \vec{V}_B - \vec{V}_A$
 $= (45 - 75)\hat{c} = -30 \hat{c}$



i.e. The bicycle seems to the driver of the car as it moves towards him with a velocity of magnitude 30 km./hr.

② A police car (A) moving on a straight road measured the relative velocity of a car (B) with respect to it moving in the opposite direction to be 120 km./hr. When the car (A) decreased its speed into the half then it remeasured the relative speed (of the car B) it found that it became 100 km./hr. What is the actual velocity of each of the two cars ?

Solution:

Assume the unit vector $\hat{c}$ in the direction of the car A.

$\therefore \vec{V}_{BA} = -120 \hat{c} \quad , \quad \therefore \vec{V}_B - \vec{V}_A = -120 \hat{c} \rightarrow (1)$

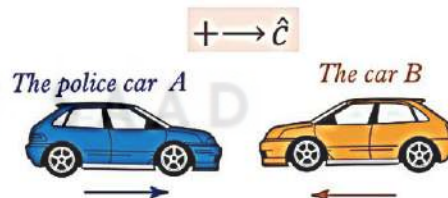
When the car A decreased its speed into the half

$\therefore \vec{V}_{BA} = -100 \hat{c} \quad , \quad \therefore \vec{V}_B - \frac{1}{2}\vec{V}_A = -100 \hat{c} \rightarrow (2)$

From (1) and (2) $\rightarrow ((1) \times -1 + (2)) \quad \therefore \frac{1}{2}\vec{V}_A = (120 - 100)\hat{c} \quad \therefore \vec{V}_A = 40\hat{c}$

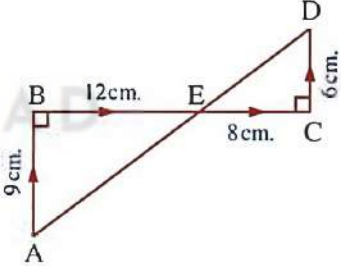
By substituting in (1) $\therefore \vec{V}_B = -80\hat{c}$

$\therefore V_A = 40 \text{ km./hr.} \quad , \quad V_B = 40 \text{ km./hr. in the opposite direction of A.}$



Exercises

[1] Choose the correct answer:

1)	If a ball is projected vertically upwards to reach a height of 3 metres and return back to the point of projection, then the magnitude of its displacement equals (a) 3 metres. (b) 6 metres. (c) zero. (d) 9 metres.	
2)	When a body moves, then the magnitude of the displacement the covered distance. (a) > (b) $\geq$ (c) < (d) $\leq$	
3)	If a body moved in a straight line 9 metres towards East, then it returned back 3 metres towards West, then its displacement = (a) 12 m East. (b) 12 m West. (c) 6 m East. (d) 6 m West.	
4)	A body moved a distance 48 metres towards East, then it changed its direction to move 20 metres towards North, then the displacement = (a) 68 m. in direction of North of the East. (b) 52 m. in direction of North of the East. (c) 68 m. in direction $22^\circ 37' 12''$ North of East. (d) 52 m. in direction $67^\circ 22' 48''$ East of North.	
5)	 A cyclist covered 6 km. towards West, then he moved afterwards 8 km. in the direction 60° South of the West, then the magnitude of the displacement that the cyclist moved = km. (a) 14 (b) $2\sqrt{37}$ (c) $7\sqrt{3}$ (d) 12.1	
6)	In the opposite figure : If each of $\overline{DC}$ and $\overline{AB}$ is perpendicular to $\overline{BC}$ If a body moved from A to B , then to C and stopped at D Then (the distance covered by the body + the magnitude of the displacement) = cm. (a) 35 (b) 70 (c) 25 (d) 60	

7)	The position vector of a body is given by the relation $\vec{r} = (t^2 - 9)\hat{i} + t\hat{j}$, then the displacement $\vec{s} = \dots\dots\dots$ (a) $t^2\hat{i} + t\hat{j}$ (b) $9\hat{i} + t\hat{j}$ (c) $(t^2 - 9)\hat{i} + t\hat{j}$ (d) $-t^2\hat{i} - t\hat{j}$
8)	In orthogonal cartesian coordinate system. If $\vec{A} = (-2, 3)$, $\vec{B} = (1, -1)$ are two position vectors of the points A and B, then the displacement from B to A = $\dots\dots\dots$ (a) $3\hat{i} + 4\hat{j}$ (b) $-3\hat{i} + 4\hat{j}$ (c) $3\hat{i} - 4\hat{j}$ (d) $-\hat{i} + 2\hat{j}$
9)	 A particle moves such that its position vector $\vec{r}$ is given as a function of time in terms of the two fundamental unit vectors $\hat{i}$ and $\hat{j}$ by the relation $\vec{r} = (6t - 3)\hat{i} + (8t + 1)\hat{j}$ then the displacement till $t = 3$ second is $\dots\dots\dots$ (a) $18\hat{i} + 24\hat{j}$ (b) $15\hat{i} + 25\hat{j}$ (c) $14\hat{i} + 18\hat{j}$ (d) $15\hat{i} + 24\hat{j}$
10)	If the position vector of a particle moves in a straight line from point (O) is given as a function of time (t) in second by the relation : $\vec{r} = (2t^2 + 3)\vec{n}$, then the magnitude of displacement $\vec{s}$ after 2 seconds equal $\dots\dots\dots$ length unit. (a) 4 (b) 6 (c) 8 (d) 11
11)	 20 m./sec. = $\dots\dots\dots$ km./hr. (a) $\frac{50}{9}$ (b) 72 (c) 0.18 (d) 20
12)	45 km./minute = $\dots\dots\dots$ m./sec. (a) 750 (b) 12.5 (c) 162 (d) 0.75
13)	 If a car moves with uniform speed 75 km./hr. for 20 minutes, then the covered distance equals $\dots\dots\dots$ km. (a) 15 (b) 20 (c) 25 (d) 30

14)	A car covered a distance 180 km. within 120 minutes , then its average speed equals km./hr. (a) 90 (b) 1.5 (c) 180 (d) 25	
15)	Two cars A and B are moving in the same straight line and the algebraic measure of their velocities were v_A and v_B respectively , then $\frac{v_A}{v_B} = \dots\dots\dots$ (a) 1 : 3 (b) 1 : 2 (c) 3 : 1 (d) 2 : 1	
16)	A car moves in a straight line , it covered a distance of 360 km. in 240 minutes , then it covered 120 km. in the same direction during 60 minutes , then its average speed = km./h. (a) 105 (b) 120 (c) 96 (d) 80	
17)	If a car covered 100 km. in the direction of East in $1\frac{1}{4}$ hour , then it covered 60 km. in the direction of West in $\frac{3}{4}$ hour , then its average velocity vector = $\vec{n}$ where $\vec{n}$ is the unit vector in the East direction and magnitude of velocity in km./hr. (a) 20 (b) 60 (c) 40 (d) 80	
18)	A body moves in a straight line 100 m. with speed 5 m./sec. then it moves with speed 8 m./sec. in the same direction for 10 sec. then the average speed during the whole trip = m./sec. (a) $\frac{2}{3}$ (b) 6 (c) 8 (d) 9	
19)	A cyclist moved 40 km. on a straight road with speed 20 km./h. in direction of a unit vector $\vec{c}$, then he covered 15 km. in the opposite direction with speed 15 km./h. , then the average velocity (in km/h) during the whole trip = (a) $7\frac{1}{2}\vec{c}$ (b) $21\frac{2}{3}\vec{c}$ (c) $\frac{55}{3}\vec{c}$ (d) $8\frac{1}{3}\vec{c}$	

20)	 A particle moves in a straight line from a fixed point (O) such that its position vector $\vec{r}$ is given by the relation $\vec{r} = (t^2 + 3t + 5) \vec{n}$ where $\vec{n}$ is a unit vector parallel to the straight line. then the average velocity after 3 seconds from the beginning of motion is (a) $20 \vec{n}$ (b) $\frac{20}{3} \vec{n}$ (c) $6 \vec{n}$ (d) $18 \vec{n}$
21)	If $\vec{v}_A = 120 \vec{c}$, $\vec{v}_B = -80 \vec{c}$, then $\vec{v}_{BA} = \dots\dots\dots$ (a) $40 \vec{c}$ (b) $-40 \vec{c}$ (c) $200 \vec{c}$ (d) $-200 \vec{c}$
22)	 A cyclist (A) moves on a straight road with velocity 15 km./hr. If another cyclist (B) moves with a velocity of 12 km./hr. on the same direction , then the algebraic measure of the velocity of B with respect to A equals km./hr. (a) 3 (b) -3 (c) 27 (d) -27
23)	 If $\vec{v}_{AB} = 15 \vec{i}$, $\vec{v}_A = 35 \vec{i}$, then $\vec{v}_B = \dots\dots\dots$ (a) $-50 \vec{i}$ (b) $-20 \vec{i}$ (c) $20 \vec{i}$ (d) $50 \vec{i}$
24)	 A and B are two bodies move in two opposite directions , if the velocity magnitude of A is twice the velocity magnitude of B , then v_{AB} equals (a) $1.5 V_A$ (b) $2 V_A$ (c) $2.5 V_A$ (d) $3 V_A$
25)	A motorcycle moves with speed 40 km./hr. in direction of a fixed unit vector $\vec{c}$, its rider watches a car , it seems to him that it moves in the opposite direction with speed 105 km./hr. , then the velocity of the car is (a) $155 \vec{c}$ (b) $-65 \vec{c}$ (c) $65 \vec{c}$ (d) $-155 \vec{c}$
26)	 The distance between two cities A and B is 120 km. A car moved from the city A towards the city B with a speed of 88 km./hr. At the same moment, another car moved from the city B towards the city A with a uniform speed of 72 km./hr. , then the two cars will meet at a distance km. from A. (a) 54 (b) 90 (c) 80 (d) 66

[2] Answer the following questions:

[1] Two cars moves at the same time from Banha towards Cairo with a constant velocity for each of them. If the velocity of the first car equals 70 km/hr and the velocity of the second car equals 84 km/hr. Find the taken time by the driver of the second car to reach the first car at the end of the trip whose length 49 km?

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[2] A train of length 150 meter entered a straight tunnel of length S meter. it took the entire crossing of the tunnel in a time of 15 seconds. Find the length of the tunnel if the train moves with uniform velocity equals 90 km/hr.

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[3] A cyclist covered 30 km on a straight road with a velocity of 15 km/hr, then he returned back and covered 10 km in the opposite direction with a velocity of 10 km/hr. Find the average velocity during his whole trip.

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[4] A traveler moved on a straight road, he covered 800 meters with velocity 9 km/hr., and then he returned back and covered the same distance in the same direction with velocity 4.5 km/hr., Find the magnitude of the average velocity of the traveler during the whole trip.

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WITH SHERIF SAAD

Lesson 2: Uniformly accelerated rectilinear motion.

The definition of the acceleration :

It is the rate of change of the velocity with respect to time or it is the change of the velocity in the unit of time.

The equations of the uniform variable motion in a straight line

The symbol	Its meaning
$\vec{v}_0$	To denote the velocity of the particle at the beginning of the measuring time.
$\vec{v}$	To denote the velocity of the particle at the end of a time interval (t)
$\vec{s}$	To denote the displacement of a particle during the time interval (t)
$\vec{a}$	To denote the acceleration.
t	time interval

$$v = v_0 + at$$

$$S = v_0 t + \frac{1}{2} at^2$$

$$v^2 = v_0^2 + 2as$$

Remarks

- ① The previous equations connect between four unknown values we can get one of them if we knew the values of the other three of them .
- ② The sign of each of v , v_0 , a and is determined according to the direction unit vector $\hat{c}$
- ③ At the beginning of motion: $t = \text{zero}$
- ④ If the body starts its motion from the rest then: $v_0 = \text{zero}$
- ⑤ If the particle reaches the maximum distance or (if the particle became at rest) then $v = \text{zero}$
- ⑥ If the particle moves with uniform velocity ((or maximum velocity)) then: $a = \text{zero}$
- ⑦ If the particle returns back to the start point (the initial position) then $s = \text{zero}$
- ⑧ In the case of knowing the values of v , v_0 , t it is not necessary to get the value of (a) to find the displacement s where we can use the equation: $s = \left(\frac{v_0 + v}{2} \right) \times 2$
i.e. ($s = v_a t$) (which we used to get the second equation)
- ⑨ The direction of the velocity is always in motion direction but the direction of acceleration could be in direction of motion (acceleration) or in opposite direction of motion (retardation).
- ⑩ Any retardation motion does not continue unless to a finite period of time , then it turns over to accelerated motion in the opposite direction .

The distance
= the average velocity $\times$ time

For Example:

① A particle moves in a straight line with a uniform acceleration 2 cm./sec^2 , in the same direction of its motion. After covering a distance 2.25 meters its velocity became 50 cm./sec. . What is its initial velocity ?

Solution:

Consider the positive direction is the direction of the motion of the particle .

$$\therefore v^2 = v_0^2 + 2as$$

$$\therefore 50^2 = v_0^2 + 2 \times 2 \times 225$$

$$\therefore v_0^2 = 50^2 - 2 \times 2 \times 225 = 1600 \quad \therefore v_0 = \pm 40$$

$$\therefore v_0 \text{ is in the positive direction} \quad \therefore v_0 \text{ (the initial velocity)} = 40 \text{ cm./sec.}$$

$$a = 2 \text{ cm./sec}^2.$$

$$s = 225 \text{ cm.}$$

$$v = 50 \text{ cm./sec.}$$

② A particle moves with uniform acceleration in a constant direction as the same direction of its initial velocity. If the particle covered a distance 100 cm. within the first 5 seconds from its motion and covered 90 cm. in the seconds 8^{th} and 9^{th} and 10^{th} from its motion. Find the acceleration also find the initial velocity.

Solution:

Consider the direction of the initial velocity is the positive direction.

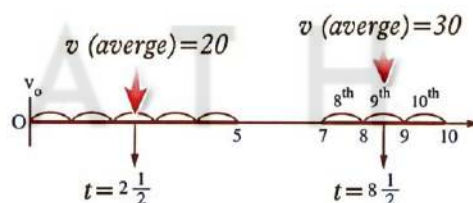
$$\therefore v = v_0 + at$$

$$\therefore 30 = 20 + a(8.5 - 2.5)$$

$$\therefore a = \frac{30-20}{8.5-2.5} = \frac{5}{3} \text{ cm./cm}^2.$$

$$\therefore v = v_0 + at$$

$$\therefore 20 = v_0 + \frac{5}{3}(2.5) \quad \therefore v_0 = 20 - \frac{5}{3}(2.5) = \frac{95}{6} \text{ cm./sec.}$$



$$v_a \text{ in the } 5 \text{ sec} \\ = \frac{100}{5} \text{ cm./sec.}$$

$$v_a \text{ in time interval} \\ \text{the seconds} \\ (8^{\text{th}}, 9^{\text{th}}, 10^{\text{th}}) \\ = \frac{90}{3} \text{ cm./sec.}$$

③ A particle moves in a constant direction with initial velocity 20 cm./sec. and a uniform acceleration 8 cm./sec^2 , in the direction of its velocity. Find :

(1) The covered distance within the 5^{th} second only.

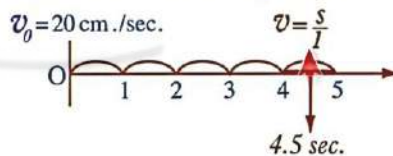
(2) The covered distance within the two seconds 7^{th} and 8^{th} together.

Solution:

(1) The average velocity within the 5^{th} sec.=the velocity after 4.5 sec. from starting motion

$$\therefore v = v_0 + at$$

$$\therefore v_a = 20 + 8(4.5) = 56 \text{ cm./sec.}$$



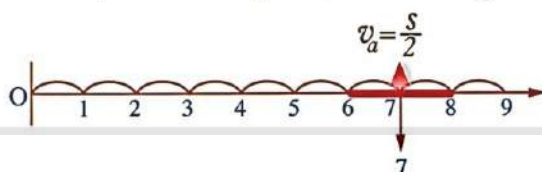
$$\therefore \text{The covered distance in the } 5 \text{ second only} = v_a t = 56 \times 1 = 56 \text{ cm.}$$

(2) The average velocity within the two seconds (7^{th} and 8^{th} together)=the velocity after 7 seconds from starting motion

$$\therefore v = v_0 + at$$

$$\therefore v_a = 20 + 8(7)$$

$$\therefore v_a = 20 + 56 = 76 \text{ cm./sec.}$$



$$\therefore \text{The covered distance in the } 5 \text{ second only} = v_a t = 76 \times 2 = 152 \text{ cm.}$$

Exercises

[1] Choose the correct answer:

1)	180 m./hr./sec. = cm./sec ² (a) $\frac{1}{20}$ (b) 5 (c) 30 (d) 300
2)	At the beginning of motion of a body , always = zero. (a) v_0 (b) v (c) t (d) a
3)	If a particle is moving with uniform velocity , then a = (a) a positive number. (b) a negative number. (c) zero. (d) v
4)	If a particle moved in a straight line , then returned to its initial position , then (a) $v = v_0$ (b) $a = 0$ (c) $s = 0$ (d) $v = 0$
5)	When a particle reaches to its maximum distance , then (a) $a = 0$ (b) $v = 0$ (c) $vt = s$ (d) $v = v_0$
6)	If the curve of the speed is represented by a horizontal segment above the time axis in the speed-time graph , then the body (a) moves with zero acceleration. (b) moves with uniform non-zero acceleration. (c) is at rest. (d) moves with increasing speed.
7)	A car is moving in a straight line from rest with uniform acceleration of magnitude 25 cm./sec ² in the same direction of the car motion , then velocity of the car after $\frac{1}{4}$ minute = (a) 375 m./sec. (b) - 3.75 m./sec. (c) 3.75 cm./sec. (d) 3.75 m./sec.
8)	An airplane increases its speed at the rate 15 m./sec ² , then the time needed to increase its speed from 100 m./sec. to 160 m./sec. is sec. (a) 17 (b) 0.05 (c) 4 (d) 1.25

9)	An airplane increases its speed at the rate 15 m./sec.^2 , then the time needed to increase its speed from 100 m./sec. to 160 m./sec. is sec. (a) 17 (b) 0.05 (c) 4 (d) 1.25
10)	A plane accelerates on a runway at a rate of 3 m./sec.^2 where taking off velocity is of magnitude 96 m./sec. , then the elapsed time from starting motion from rest to the taking off = sec. (a) 25 (b) 32 (c) 37 (d) 42
11)	A particle started its motion with velocity 24 m./sec. and with retardation 8 m./sec.^2 , then the particle stopped after sec. (a) 3 (b) 16 (c) $\frac{1}{3}$ (d) 30
12)	 The distance which a particle covered in a constant direction from rest with an acceleration of magnitude 5 cm./sec.^2 during a time of magnitude 4 seconds = cm. (a) 10 (b) 20 (c) 40 (d) 80
13)	If a particle started its motion with velocity 30 cm./sec. and with uniform acceleration 5 cm./sec.^2 in the same direction of its initial velocity, then the covered distance after 10 seconds. from starting motion = cm. (a) 550 (b) 300 (c) 750 (d) 1500
14)	If a particle moves in a straight line with uniform acceleration, then $v + v_0 =$ (a) $\frac{2s}{t}$ (b) $\frac{s}{t}$ (c) $\frac{s}{2t}$ (d) $2st$
15)	 A particle moves from rest in a straight line with uniform acceleration. it covered 24 m. in the first four seconds of starting its motion, then the magnitude of its acceleration = m./sec.^2 (a) 3 (b) 6 (c) 12 (d) $\frac{3}{2}$

16)	A ball is projected horizontally against direction of wind with initial speed $v_0 = 15 \text{ cm./sec.}$ to move with uniform deceleration 5 cm./sec.^2, then the time elapsed till the ball returned to its initial position = sec. (a) 2 (b) 3 (c) 4 (d) 6
17)	A particle started its motion from rest in a straight line with uniform acceleration 3 cm./sec^2. It covered a distance 24 cm., then its velocity at the end of this distance = cm./sec. (a) 144 (b) 12 (c) 24 (d) 72
18)	 A car moved from rest with acceleration of magnitude 4 m./sec^2, then the distance that the car covered till its speed becomes 24 m./sec. is m. (a) 72 (b) 35 (c) 96 (d) 81
19)	 A racing car moves on the track with speed 44 m./sec. in a certain moment then its speed decreases with a constant rate until it becomes 22 m./sec. in 11 seconds, then the distance that the car covered within this time = m. (a) 238 (b) 415 (c) 363 (d) 348
20)	 The average velocity of a particle moving with initial velocity (v_0) and a uniform acceleration (a) within the 6th second from the beginning of motion = (a) $v_0 + 5a$ (b) $v_0 + 6a$ (c) $v_0 + 5\frac{1}{2}a$ (d) $v_0 + 3a$
21)	 The average velocity of a particle moving with initial velocity (v_0) and a uniform acceleration (a) within (7th, 8th and 9th) seconds from the beginning of motion = (a) $v_0 + 7\frac{1}{2}a$ (b) $v_0 + 8a$ (c) $v_0 + 8\frac{1}{2}a$ (d) $v_0 + 9a$

22)	A body starts its motion in a constant direction with initial velocity 15 cm./sec. and uniform acceleration of magnitude 4 cm./sec² in the same direction of its initial velocity , then the covered distance during the 6th second only = cm. (a) 32 (b) 35 (c) 39 (d) 37	
23)	A particle starts its motion in a straight line with initial velocity 5 m./sec. and uniform acceleration 2 m./sec² in the same direction of its initial velocity , then the covered distance during 6th , 7th and 8th seconds only = m. (a) 18 (b) 54 (c) 36 (d) 57	
24)	 A body moves in a straight line with uniform acceleration. It covered 52 metres in the first four seconds , then it covered a distance 92 metres in the next four seconds. , then the acceleration of motion = m./sec². (a) - 2.5 (b) 3 (c) 3.5 (d) 2.5	
25)	A body moves in a straight line with uniform acceleration. It moves 10 m. in the 1st second and 15 m. in the 2nd second , then the distance covered during the 3rd second = m. (a) 20 (b) 25 (c) 30 (d) 45	

[2] Answer the following questions:

[1] A car moves from rest with acceleration of magnitude 4 m/sec². What is the distance that the car covered when its velocity became 24 m/sec?

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[2] A racing car moves in the track with velocity 44 m/sec then its velocity decreases with a constant rate until it becomes 22 m/sec through 11 seconds. Find the distance that the car covered through that time .

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[3] A particle moves in a straight line with a uniform acceleration so its velocity increased from 15 m/sec into 25 m/sec. after covering 125 meter .Find the time takes for that .

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[4] A cyclist moves with a uniform acceleration until its velocity became 7.5 m/sec through 4.5 seconds. If the displacement of the bicycle through the accelerating interval equals 19 meters, find the initial velocity for the bicycle.

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[5] Karim practices on riding the bicycle. His father pushes him to gain a constant acceleration of magnitude $\frac{1}{2} \text{ m / sec}^2$ for 6 seconds and after that Karim rides the bicycle alone with the velocity gained for another 6 seconds before he falls on the ground. Find the distance that Karim will cover.

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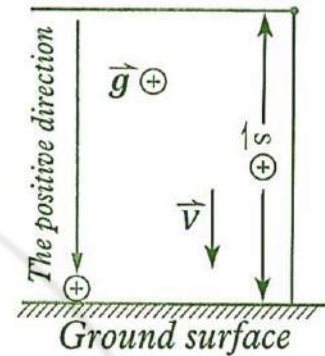
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Lesson 3: Vertical motion under the effect of gravity (Free fall).**Laws of vertical motion of bodies:****First: If the body is falling or projected downwards**

$$\begin{aligned}
 v &= v_0 + gt \\
 s &= v_0 t + \frac{1}{2}gt^2 \\
 v^2 &= v_0^2 + 2gs
 \end{aligned}$$

The position of falling or projection a body downwards

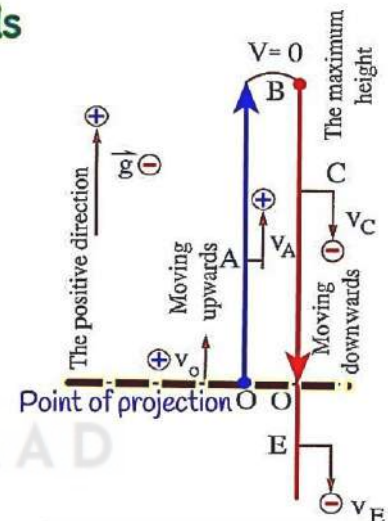
**Remarks**

We consider the vertical direction downwards is the positive direction. In this case each of v_0 , v , g and s are positive and hence :

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|---|---|
| ① | Each of (v) and (s) increases by increasing time (t) measured from the moment of falling or projecting downwards and also (v) increases whenever (s) increases from the point of falling or projecting downwards. |
| ② | The displacement (s) at any time interval is equal to the distance covered during this interval. |
| ③ | If a body fell down (i.e. It begins its motion from rest) then $v_0 = 0$ |

Second: If the body is projected vertically upwards

$$\begin{aligned}
 v &= v_0 - gt \\
 s &= v_0 t - \frac{1}{2}gt^2 \\
 v^2 &= v_0^2 - 2gs
 \end{aligned}$$



Remarks

In this case, we take the upwards vertical direction is a positive direction thus v_0 is positive g is negative.

①	If a body is projected from the point (O) vertically upwards then its velocity decreases till it becomes zero at the point B, we then say that the body reaches its (maximum height) which is OB then the body changes its direction descending from rest and its acceleration becomes positive and its velocity increases until it comes back to (O). If it is not at rest at (O) then it will continue in falling vertically downwards as in the shown figure.
②	The velocity of the body during the moving upwards is positive and during moving downwards is negative.
③	The displacement (s) is positive if it is in the positive direction.
④	The displacement in any interval of time <u>may not be equal</u> to the covered distance during this interval.
⑤	The velocity of the body at any point when it is ascending is equal to its velocity at the same point when it is descending with change in the direction of the two velocities.
⑥	Notice that when a body is projected vertically upwards then it moves on the vertical line passing through the point of projection and also returns back on the same vertical line but as solving problems it is better to draw the descending line beside the ascending line for illustration as shown in the preceding figure.
⑦	t (the time of reaching maximum height) = $\frac{\text{The magnitude of the initial velocity } (v_0)}{\text{The magnitude of acceleration of gravity } (g)}$
⑧	s (the maximum height) = $\frac{\text{The square of magnitude of the initial velocity } (v_0)^2}{\text{Twice the magnitude of acceleration of gravity } (2g)}$

Rule: If a body is projected vertically upwards, then :

(1)	The time taken to reach the maximum height equals the time of moving downwards to reach the point of projection.
(2)	The velocity with which the body returns back to the point of projection = $-$ (the velocity of projection)

WITH SHERIF SAAD

① From the top of a tower of height 112 meters. A body is projected downwards with velocity 8.4 m./sec. Calculate :

- (1) The distance which the body covered during the third second of its motion.
- (2) Time taken to reach the ground surface.
- (3) The velocity of the body at the moment of reaching the ground surface.

Solution: Consider the positive direction is the vertical direction downwards.

(1)	The average velocity of the body during the 3rd second = the velocity of the body after 2.5 seconds from beginning motion. $\therefore v = v_0 + gt \quad \therefore v = 8.4 + 9.8 \times 2.5 = 32.9 \text{ m./sec.}$ <math>s \text{ (the covered distance during the 3rd sec.)} = v_A t = 32.9 \times 1 = 32.9 \text{ m.}</math>	$v_0 = 8.4 \text{ m./sec.}$ $g = 9.8 \text{ m./sec.}$
(2)	$\therefore s = v_0 t + \frac{1}{2}gt^2 \quad \therefore 112 = 8.4t + \frac{1}{2}9.8t^2 \quad (\div 0.7)$ $\therefore 7t^2 + 12t - 160 = 0$ $\therefore (7t + 40)(t - 4) = 0 \quad \therefore t = 4 \text{ or } t = -\frac{40}{7} \text{ refused}$ $\therefore t = 4 \text{ seconds (which is the time taken to reach the ground surface)}$	$s = 112 \text{ meters.}$
(3)	$\therefore v = v_0 + gt \quad \therefore v = 8.4 + 9.8 \times 4 = 47.6 \text{ m./sec.}$ Other solution $\therefore v^2 = v_0^2 + 2gs \quad \therefore v^2 = (8.4)^2 + 2 \times 9.8 \times 112$ $\therefore v = \sqrt{(8.4)^2 + 2 \times 9.8 \times 112} = 47.6 \text{ m./sec.}$	

② A body is projected vertically upwards from a point on the ground. It comes back to this point after 6 sec. . Find its initial velocity and maximum height of the body and its velocity after 4.5 sec.

Solution: Consider the (positive) direction is the vertical direction upwards.

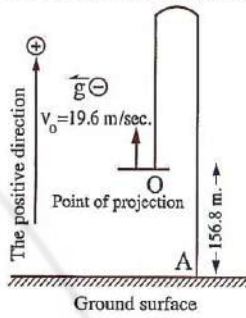
(1)	$\therefore$ The body came back to the point of projection after 6 seconds from the moment of projection. $\therefore$ Time of ascending = time of descending = $\frac{6}{2} = 3 \text{ sec.}$ $\therefore$ Time taken to reach maximum height = $\frac{v_0}{g} \quad \therefore 3 = \frac{v_0}{g}$ $\therefore v_0 = 3g = 3(9.8) = 29.4 \text{ m./sec.}$
(2)	$\therefore s = \frac{v_0^2}{2g} = \frac{(29.4)^2}{2(9.8)} = 44.1 \text{ m. (the maximum height)}$ $\therefore v = v_0 - gt \quad \therefore v = 29.4 - (9.8)(4.5) = -14.7 \text{ m./sec.}$ $\therefore$ The velocity of the body after 4.5 sec. = 14.7 m./sec. downward

③ A small stone is projected vertically upwards with velocity 19.6 m./sec. from the top of a tower of height 156.8 meters from ground surface. Find :

(1) The time elapsed from the moment of projection till it reach the ground surface.

(2) The velocity of the body when reaching the ground surface.

Solution: Consider the (positive) direction is the vertical direction upwards.

when the stone reach the ground surface then : $s = 156.8$ metre $\therefore s = v_0 t - \frac{1}{2} g t^2 \quad \therefore -156.8 = (19.6)t - \frac{1}{2}(9.8)t^2$ (1) $\therefore (4.9)t^2 - (19.6)t - 156.8 = 0 \quad (\div 4.9)$ $\therefore t^2 - 4t - 32 = 0$ $\therefore (t + 4)(t - 8) = 0 \quad \therefore t = -4 \text{ refused or } t = 8 \text{ sec.}$ $\therefore t \text{ (time of reaching ground surface)} = 8 \text{ sec.}$	
(2) $\therefore v = v_0 - g t \quad \therefore v = 19.6 - (9.8)(8) = -58.8 \text{ m./sec.}$ $\therefore v \text{ (the velocity of reaching the ground surface)} = 58.8 \text{ m./sec. downward}$	

④ A stone fell from rest from a point at height 10 meters above a hill of sand to embed in the sand a distance 1.96 m. Find the acceleration with which the stone moved in the sand.

Solution: Consider the (positive) direction is the vertical direction upwards.

Before embedding in the sand :

$$\therefore v^2 = v_0^2 + 2gs \quad \therefore v^2 = 0^2 + 2(9.8)(10) = 196$$

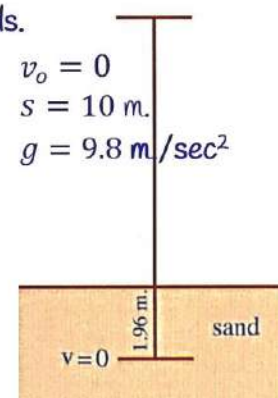
$$\therefore v = \sqrt{196} = 14 \text{ m./sec.}$$

After embedding in the sand :

$$v_0 = 14 \text{ m./sec.}, \quad s = 1.96 \text{ m.} \quad \therefore v = 0 \text{ m./sec.}$$

$$\therefore v^2 = v_0^2 + 2as \quad \therefore 0 = 14^2 + 2(a)(1.96)$$

$$\therefore a = \frac{-14^2}{2(1.96)} = -50 \text{ m./sec.}^2$$



⑤ A small ball is projected vertically upwards from a window of a house. It was seen during its returning back in front of the window after 8 seconds from the moment of projection then it reached ground surface after 9 seconds from the moment of projection. Find the height of the window over the ground surface in meters.

Solution:

$$\therefore \text{Time of the maximum height} = \frac{8}{2} = 4 \text{ seconds.}$$

$$\therefore v_0 = g t = 9.8 \times 4 = 39.2 \text{ m./sec.}$$

$$\therefore s = v_0 t - \frac{1}{2} g t^2$$

$$\therefore s = 39.2 \times 9 - \frac{1}{2} \times 9.8 \times 9^2$$

$$= -44.1 \text{ m.}$$

i.e. The window is at a height 44.1 meters over the ground surface.

⑥ A ball fell vertically downwards from a height 2.5 m. towards horizontal ground to impinge with it then it rebounds vertically upwards with velocity of magnitude $= \frac{4}{5}$ the magnitude of its velocity before it impinges ground directly. Find the maximum height it reaches to it after its impinging the ground for the first time.

Solution:

• In the case of descending :

$$\therefore v^2 = v_0^2 + 2gs$$

$$v^2 = 0 + 2(9.8)(2.5) = 49$$

$$\therefore v = 7 \text{ m./sec.}$$

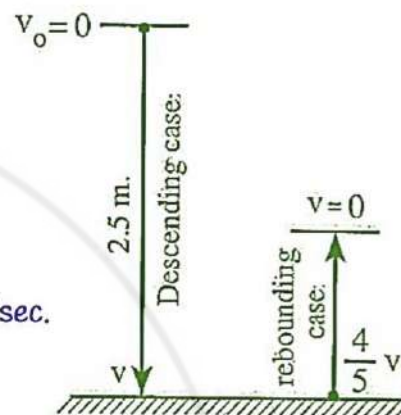
$\therefore v$ (the velocity of the ball before it impinges the ground directly) = 7 m./sec.

• In the case of rebounding :

Consider the (positive) direction is the vertical upwards

$$\therefore v_0 = \frac{4}{5} \times 7 = 5.6 \text{ m./sec.}$$

$$\therefore \text{The maximum height} = \frac{v_0^2}{2g} = \frac{(5.6)^2}{2(9.8)} = 1.6 \text{ m.}$$



Exercises

[1] Choose the correct answer:

1)	If a body fell vertically downwards , then	
	(a) $v = v_0$ (b) $v = \text{zero}$ (c) $v_0 = 0$ (d) $s = vt$	
2)	When the body reaches its maximum height , then = zero	
	(a) s (b) v (c) v_0 (d) a	
3)	If the body projected upward is returned to the point of projection , then	
	(a) $ v = v_0 $ (b) $s = 0$ (c) $a = 0$ (d) a and b	
4)	If a body is projected up with speed (v_0) m./sec. to reach its maximum height (s) m. the time needed to reach the maximum height equals seconds.	
	(a) $\frac{v_0}{2g}$ (b) $\frac{v_0}{g}$ (c) $\frac{v_0^2}{2g}$ (d) $\frac{2 v_0}{g}$	

5)	A body is projected vertically up with velocity 98 m./sec. , then the time needed to reach maximum height = sec. (a) 15 (b) 10 (c) 3 (d) 20
6)	A body is projected vertically upwards with velocity 42 m./sec. , then the maximum height which the body reaches equals (a) 65 m. (b) 98 m. (c) 84 m. (d) 90 m.
7)	A body is projected vertically upward from a point on the ground surface , if the maximum height the body reached is 44.1 m. , then the velocity of projection = m./sec. (a) 44.1 (b) 22.05 (c) 19.6 (d) 29.4
8)	A body is projected vertically upward to cover a distance 88.2 m. till it returned to the projection point , then the taken time = sec. (a) 4 (b) 6 (c) 5.5 (d) 3
9)	If a body is projected vertically up from the ground surface and it returns back to the point of projection after 12 seconds , then the falling time = seconds. (a) zero (b) 3 (c) 6 (d) 12
10)	If a body is projected vertically up from a point on the ground surface and it returns back to the same point after 10 seconds , then its maximum height the body can reached = m. (a) 122.5 (b) 245 (c) 49 (d) 490
11)	If a bullet fired vertically upward from a point on the ground surface and returned to the projection point after 10 seconds , then the initial speed of the bullet = m./sec. (a) 9.8 (b) 49 (c) 25 (d) 98
12)	A body is projected vertically down with velocity 15 m./sec. from a height 100 m. , then its velocity after 3 seconds = m./sec. (a) 46.7 (b) 29.4 (c) 44.4 (d) 32.8

13)	A ball fell vertically downward from the top of a tower of height 122.5 m. , then the elapsed time to reach ground surface = sec. (a) 4 (b) 7 (c) 6 (d) 5
14)	A body is projected vertically downward with velocity 32 m./sec. , it reached the ground surface after 4 seconds , then the height from which the body is projected = m. (a) 206.4 (b) 105.7 (c) 198.2 (d) 217
15)	A body is projected vertically downward , it covered a distance 15.5 m. during the 1 st second from the projection , then the velocity of projection = m./sec. (a) 15.5 (b) 10.6 (c) 9.8 (d) 17.2
16)	A body is projected vertically upward with velocity 49 m./sec. , then the covered distance during the 4 th second only from the moment of projection = m. (a) 9.8 (b) 14.7 (c) 117.6 (d) 24.5
17)	A stone is projected vertically upward with velocity 28 m./sec. from the ground surface , it fell on the top of a house after 4 second from the moment of projection : then the height of the house = m. (a) 112 (b) 98 (c) 33.6 (d) 38.4
18)	A helicopter is moving vertically upward with uniform velocity 10 m./sec. , a parachutist fell vertically downward from it , then the initial velocity of the parachutist = (a) zero (b) 10 m./sec. downward. (c) 10 m./sec. upward. (d) 9.8 m./sec downward.
19)	A balloons moving vertically upward with velocity 24.5 m./sec. , a body fell down from it to reach ground surface after 8 seconds , then height of balloons at the moment of falling the body from the ground surface = m. (a) 117.6 (b) 119.8 (c) 108.7 (d) 96.8

Answer the following questions:

① A child throws a ball from a window that rises 3.6 m from the pavement. What will be its velocity at the moment of contact with the pavement?

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② A ball fell vertically downwards. What is its velocity after 6 seconds from the moment of its falling?

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③ A body fell vertically downwards from height 490 m from the surface of the ground find:

(a) Time of reaching the ground surface.

(b) Its velocity after 5 seconds from starting the motion.

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④ A rubber ball fell from a height of 10 meters so it hit the ground and rebounded vertically upward a distance $2\frac{1}{2}$ meters. Calculate the velocity of the ball just after and before hitting the ground.

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- ⑤ A student practices on kicking football vertically upwards in air, then the ball returns due to the impact of every kick. So, it hits his foot. If the ball takes from the moment of its kicking until colliding with his foot 0.3 seconds.

(a) Find the initial velocity.

(b) The height that the ball reaches after the student kicked it.

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- ⑥ A body is projected vertically upwards from the top of a hill of height 9.8 meters with velocity 4.9 m/sec Find:

(a) Velocity of the body at the moment of reaching the bottom of the hill.

(b) The time taken to reach the bottom of the hill.

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- ⑦ A stone is projected in a well with velocity 4 m/sec vertically downwards so, it reached the bottom of the well after 2 seconds. Find:

(a) The depth of the well.

(b) Velocity of the stone when it collides with bottom of the well.

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Unit 2:

Newton's Laws of Motion

Lesson 4: Momentum

Lesson 5: Newton's first law

Lesson 6: Newton's second law

Lesson 7: Newton's third law

Lesson 8: Follow applications on Newton's law

WITH SHERIF SAAD

Lesson 4: Momentum

The mass

- The mass of the body is a scalar quantity directly proportional to the weight of this body.
- The mass is denoted by m and from its measuring units: ton, k g. or gm.
- **One ton** = 1000 kilogram., **one kilogram** = 1000-gram,
one gram = 1000 milligram

The momentum

The momentum vector of a particle, denoted by $\vec{H}$ is defined as the product of the mass of the particle and its velocity vector.

- If the mass of the particle is m , the velocity vector at an instant $= \vec{v}$, then the momentum of the particle at this instant is $\vec{H} = m \vec{v}$
- It is clear to us now from the definition that the momentum of a particle at an instant is a vector having the same direction of its velocity at this instant.
- If the motion is on a straight line, then each of $\vec{H}$ and $\vec{v}$ is parallel to the straight line of motion, according to this we can express each of these two vectors in terms of the algebraic measure of them related to a constant unit vector $\hat{c}$ parallel to this straight line, then we write :

$$\vec{H} = H \hat{c}, \vec{v} = v \hat{c} \quad \therefore H \hat{c} = m v \hat{c} \quad \text{i.e. } H = m v$$

The change in momentum

If $\vec{H}_1$ is the momentum vector of a particle of mass m_1 and with velocity $\vec{v}_1$ at a certain instant (t_1) and $\vec{H}_2$ is the momentum vector of the same particle after its mass became m_2 and its velocity became $\vec{v}_2$, at the instant (t_2) then :

$$\text{The change in momentum vector : } \Delta \vec{H} = \vec{H}_2 - \vec{H}_1 = m_2 \vec{v}_2 - m_1 \vec{v}_1$$

$$\text{If the mass is constant, then : } \Delta \vec{H} = m (\vec{v}_2 - \vec{v}_1) \quad , \quad \Delta H = m (v_2 - v_1)$$

- Magnitude of the change in momentum vector $= \|\vec{H}_2 - \vec{H}_1\|$

1)

 Calculate the momentum of a car whose mass is 800 kg. moving towards the Western South with a uniform speed of a magnitude 126 km./h.

Sol.

The momentum of the car

$$= 800 \times 126 = 100800 \text{ kg.m. km/hr}$$

in direction of western south.

2)

A body of mass 7.5 grams , moves from rest in a straight line with uniform acceleration 9 cm./sec.^2 in the direction of its motion. Calculate its momentum after $\frac{1}{2}$ minute from the moment of the beginning of motion.

Sol.

$$v = v_0 + at = 0 + 9 \times 30 = 270 \text{ cm./sec.}$$

$$H = \frac{75}{10} \times 270 = 2025 \text{ gm.cm./sec.}$$

3)

Calculate the momentum of a stone of mass 250 gm. when it falls a distance 4.9 m. vertically downwards.

Sol.

4)

A car of mass 2 tons moves from rest with a uniform acceleration 150 cm./sec^2 Find :

(1) Its momentum after 10 seconds from the moment of beginning motion.

(2) The change of its momentum during the fifth second.

Sol.

Let $\hat{C}$ be unit vector in direction of motion

$$\begin{aligned}(1) \therefore v &= v_0 + at = 150 \times 10 \\ &= 1500 \text{ cm./sec.} \\ &= 15 \text{ m./sec.}\end{aligned}$$

$$\begin{aligned}\therefore \text{The momentum} &= 2 \times 1000 \times 15 \\ &= 30000 \text{ kg.m/sec.}\end{aligned}$$

$$\begin{aligned}(2) \therefore v \text{ (after 4 seconds)} &= 150 \times 4 = 600 \text{ cm./sec.} \\ &= 6 \text{ m./sec.}\end{aligned}$$

$$\begin{aligned}v \text{ (after 5 seconds)} &= 150 \times 5 = 750 \text{ cm./sec.} \\ &= 7.5 \text{ m./sec.}\end{aligned}$$

$$\begin{aligned}\therefore \text{The change in momentum during the fifth second} \\ &= 2000 (7.5 - 6) = 3000 \text{ kg.m/sec.}\end{aligned}$$

5)

 A train wagon of a mass 15 tons moves horizontally with velocity of magnitude 40 m./sec to collide with a barrier at the end of the railroad , then it rebounds back with velocity 30 m./sec . Calculate the change of its momentum.

Sol.

Let $\hat{C}$ be unit vector in direction of rebounding

$$\begin{aligned}\therefore \text{The change of momentum} &= m (v - v_0) \\ &= 15000 (30 - (-40)) = 1050000 \text{ kg.m/sec.}\end{aligned}$$

Exercises

[1] Choose the correct answer:

1)	A car of mass 3 tons moves on a straight line with velocity 72 km./h. , then the momentum of the car equals kg.m./sec. (a) 60 (b) 60000 (c) 216 (d) 216000
2)	 The momentum of a bullet whose mass is 100 gm. moving at velocity 240 m./sec. is (a) 24×10^{-3} gm.m./sec. (b) 24 kg.m./sec. (c) 24×10^4 gm.m./sec. (d) 24×10^3 kg.m./sec.
3)	 (<i>1st session 2017</i>) : The momentum of a car whose mass is 2 tons moving in a straight line at velocity 54 km./h is (a) 108 tons.m./sec. (b) 30 tons.km./hr. (c) 30000 kg.m./sec. (d) 108000 kg.m./sec.
4)	 A body of a mass 500 gm. is let to fall from a height of 4.9 metres on the ground surface , then its momentum as it comes the ground is (a) 2.45 kg.m./sec. (b) 4.9 kg.m./sec. (c) 2450 kg.m./sec. (d) 4900 kg.m./sec.
5)	If a body of constant mass (m) moves with acceleration (a) , then its momentum (a) decreases. (b) increases. (c) remains constant. (d) insufficient givens.
6)	A body of mass 500 gm. is released from a height (h) above the ground , it has magnitude of momentum as it reach the ground surface = 8 400 gm. m/sec. , then the height h = m. (a) 28.8 (b) 57.6 (c) 14.4 (d) 16.8

7)	 A missile of mass 1 kg. is launched at speed 720 km./h. towards a tank of mass 50 tons moving towards the mortar at speed 20 m./sec. , then First : (2nd session 2017) : The magnitude of the momentum of the missile with respect to the tank is (a) 200 kg.m./sec. (b) 220 kg.m./sec. (c) 10^7 kg.m./sec. (d) 1.1×10^7 kg.m./sec. Second : The magnitude of momentum of the tank with respect to the missile is (a) 200 kg.m./sec. (b) 220 kg.m./sec. (c) 10^7 kg.m./sec. (d) 1.1×10^7 kg.m./sec.
8)	A stone of mass 20 gm. is released to fall vertically downwards , then magnitude of its momentum after 3 seconds from the instant of falling in gm.cm./sec. equals knowing that it does not reach the ground. (a) 588 (b) 600 (c) 5 880 (d) 58 800
9)	A rubber ball of mass 50 gm. is released from height 4.9 metres on horizontal ground after impact with the ground it rebounds upwards to height of 2.5 metres , then the magnitude of the change of its momentum due to the impact = gm.cm./sec. (a) 8.4 (b) 8 400 (c) 84 000 (d) 840 000

Answer the following questions:

1)

A car of mass 2 tons moves with velocity 90 km./h. Find its momentum in kg.m/sec.

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2)

A ball of mass 15 gm. moves horizontally with speed 2.52 km/hr.
If it hit a racket at rest to rebound with speed 40 cm/sec.
Find the change in the momentum of the ball.

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3)

A body fell vertically down from rest , from a height 16.9 metre above the ground
, its momentum magnitude became 5460 gm.m/sec. before hitting the ground directly.
Calculate the mass of the body. If this body rebounded after hitting the ground
vertically upwards at a height 4.9 m. Calculate the magnitude of the change of the
momentum of the body due to this impact.

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4)

A sphere of mass $\frac{3}{8}$ kg. is projected vertically upwards with speed 7 m/sec.
from a point below the ceiling of the room at a distance (1.6) metre.
It impinges the ceiling , then it rebounded downwards.
If the magnitude of change in the magnitude of its momentum due to this impact
with the ceiling equals 2400 gm.m/sec.
Calculate the velocity of rebounding of the sphere.

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Lesson 5: Newton's first law

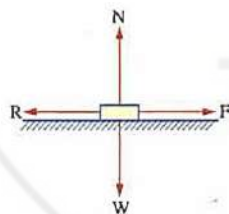
Every body preserves its state of rest or moving uniformly unless acted upon by an unbalanced external force by an external effect change that state.

- The uniform motion : It is the motion with constant velocity in magnitude and direction.
- The law considers the two states of rest and uniform motion is an equivalent status as both represent the natural state of the body when the resultant of the acting forces is equal to zero.
- This law indicates that any object is unable to change its state from rest or moving uniformly in a straight line which is known by the inertia principle.
- The resistance of the surface of motion on which the body moves is always parallel to the surface in the opposite direction of motion of the body.
- The weight of the body (W) which moves on an inclined plane , making an angle θ with the horizontal , will be resolved into two components "in the direction of greatest slope and the perpendicular to it" which are : ($W \sin \theta$) , ($W \cos \theta$)

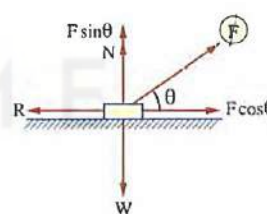
The uniform motion of some bodies

Consider a body of weight ($\vec{W}$) moves under the effect of a force ($\vec{F}$) and resistance ($\vec{R}$)

1. The uniform motion on a horizontal plane:

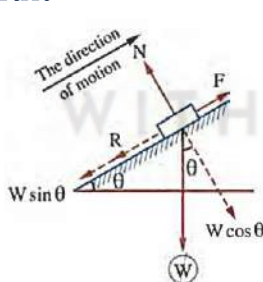


$$F = R, N = W$$

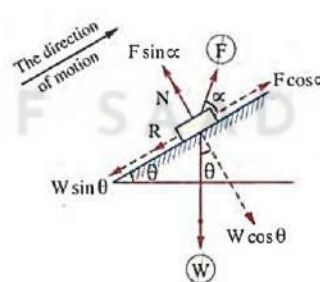


$$F \cos \theta = R, N + F \sin \theta = W$$

2. The uniform motion upwards of an inclined plane making an angle of measure θ at the horizontal:

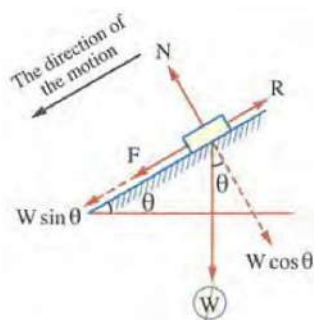


$$F = W \sin \theta + R, N = W \cos \theta$$



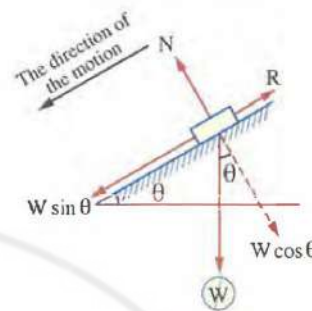
$$N + F \sin \alpha = W \cos \theta, F \cos \alpha = R + W \sin \theta$$

3. The uniform motion downwards of an inclined plane making an angle of measure θ at the horizontal:



$$F + W \sin \theta = R$$

$$N = W \cos \theta$$

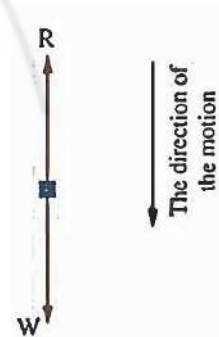


$$R = W \sin \theta$$

$$N = W \cos \theta$$

4. The uniform motion is vertical:

- If a body of weight W moved inside a liquid, then it will be subject to a resistance of magnitude (R)
 $\therefore W = R$
- This is the same as the uniform motion of a parachutist descending with his parachute where the weight of the soldier and his parachute $= W$, the magnitude of resistance of air $= R$



Remarks

- (1) If the body moves with the maximum velocity, that means the motion is uniform.
- (2) If the car stopped its engine, then F (the engine force) = zero
- (3) The total resistance = the resistance per ton $\times$ mass in tons
- (4) In case of the motion of a helicopter vertically, then the direction of the force ($\vec{F}$) is always upwards in the two cases, ascending or descending.
- (5) If a body moves under effect of a resistance of magnitude (R) proportional directly as speed (v)

i.e. $R \propto v$, then $R = k \cdot v$ where k is a constant, $\frac{R_1}{R_2} = \frac{v_1}{v_2}$

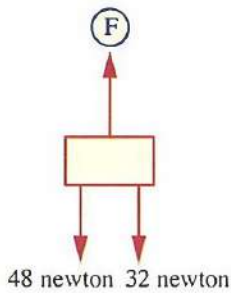
- (6) If a body moves under effect of a resistance of magnitude (R) proportional directly as the square of speed (v)

i.e. $R \propto v^2$, then $R = k v^2$ where k is a constant, $\frac{R_1}{R_2} = \frac{v_1^2}{v_2^2}$

Assuming the direction of motion is the positive direction so the algebraic measure of the velocity = the speed

1) In each of the following figures, the body is at rest under the action of a system of forces, determine the magnitudes of the forces F and K :

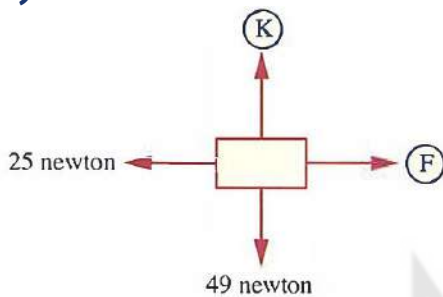
1)



$\therefore$ body at rest :

Sol. $F = 32 + 48 = 80 \text{ N}$

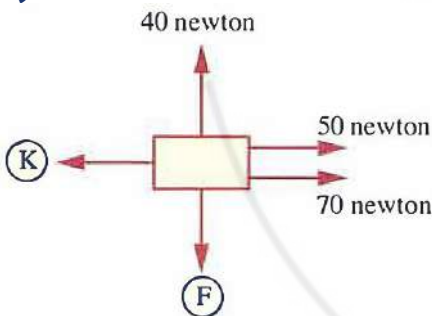
2)



$\therefore$ body at rest :

Sol. $\therefore F = 25 \text{ N} , K = 49 \text{ N}$

3)

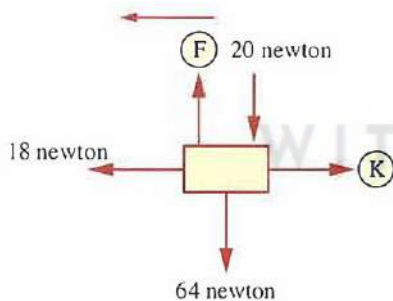


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Sol.

2) In each of the following figures, the body moves with a uniform velocity (V) under the action of a system of forces, determine the magnitudes of the forces F and K:

1)

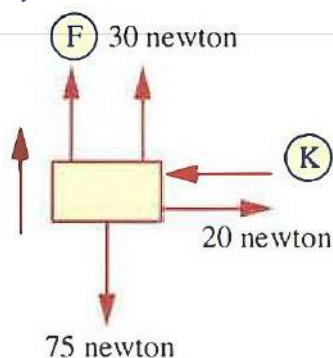


$\therefore$ The body moves by uniform velocity

$\therefore F = 20 + 64 = 84 \text{ N}$

Sol. $K = 18 \text{ N}$

2)



Sol.

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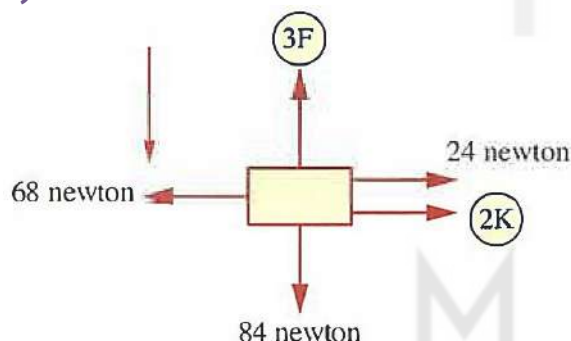
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3)



Sol.

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3)

A lorry car of weight 3 ton.wt. ascends with uniform velocity on a plane inclined to the horizontal at an angle of measure θ where $\sin \theta = \frac{1}{60}$. If the magnitude of the resistance of the friction and etc. equals 4 kg.wt/ton of the mass of the car. Find in kg.wt. the force magnitude of the car engine.

Sol.

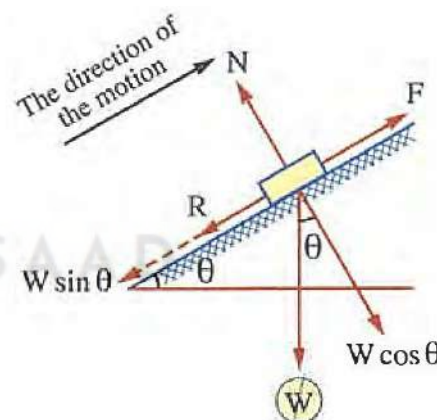
R (the magnitude of the total resistance of the car)
 $= 4 \times 3 = 12 \text{ kg.wt.}$

$\therefore$ The car ascends up the inclined plane with uniform velocity

$\therefore F = R + W \sin \theta$

$\therefore F$ (the magnitude of the force of the car engine)

$$= 12 + 3 \times 1000 \times \frac{1}{60} = 62 \text{ kg.wt.}$$



Exercises

[1] Choose the correct answer:

1)	A car of mass 3 tons moves in uniform motion on a horizontal road. If the resistance magnitude of motion is 75 kg.wt. per ton of mass , then the force magnitude of motor of the car = kg.wt. (a) 225 (b) 405 (c) 675 (d) 2205
2)	A car of mass 1200 kg. moves on a horizontal road with a uniform velocity. If the force magnitude of the engine equals 1200 newton , then the magnitude of the resistant to the motion per each ton from the mass = (a) 1 newton. (b) 9.8 kg.wt. (c) 1000 newton. (d) 1000 kg.wt.
3)	A tank moves with a uniform velocity on a horizontal road against resistance of magnitude equals 90 kg.wt. per ton of its mass , if the magnitude of force of the motor of the tank = 4500 kg.wt. , then the mass of the tank = ton. (a) 49 (b) 50 (c) 196 (d) 490
4)	A small metallic ball of weight 130 gm.wt. in a liquid , it is found that it moves equals distances in equal times , then the magnitude of the resistance of the liquid for motion of the ball = gm.wt. (a) 65 (b) 130 (c) 260 (d) 32.5
5)	A parachutist descends vertically with a uniform velocity. If the total weight of the parachutist and the parachute is 85 kg.wt. , then the magnitude of air resistance for the parachute = kg.wt. (a) zero (b) 8.67 (c) 85 (d) 833
6)	A horse pulls a wooden mass on horizontal ground with force of magnitude 100 kg.wt. and inclined at 30° to the horizontal. If this mass moves with a uniform velocity , then the magnitude of the ground resistance to its motion = kg.wt. (a) 50 (b) $50\sqrt{3}$ (c) 100 (d) $100\sqrt{3}$

7)	 If a body of mass 20 kg.wt. slides with a uniform velocity on an inclined plane to the horizontal with angle of measure 30°, then the resistance magnitude of the plane equals kg.wt. (a) zero (b) 10 (c) $10\sqrt{3}$ (d) 20
8)	 A parachutist lands vertically downwards and magnitude of the air resistance to its motion is directly proportional to the square of his velocity and v_1 is his velocity when the air resistance magnitude is equivalent to $\frac{9}{25}$ of his weight and v_2 is the landing maximum velocity to the parachutist. Then $v_1 : v_2$ (a) 9 : 25 (b) 25 : 9 (c) 3 : 5 (d) 5 : 3
9)	A locomotive drags a train on a horizontal road with uniform velocity, the mass of the locomotive and the train together is 250 tons and the engine force magnitude is 2000 kg.wt., then the resistance magnitude in kg.wt. for each ton of the mass is (a) $\frac{1}{8}$ (b) 8 (c) 200 (d) 250
10)	A car of mass (m) tons ascends by its maximum speed an inclined plane which inclines to the horizontal at an angle of measure 30°, if the magnitude of the force of the car engine magnitude = $\frac{3}{4}$ of its weight, then the resistance per each ton of its weight = kg.wt. (a) 25 (b) 250 (c) 500 (d) 1000

WITH SHERIF SAAD

Answer the following questions:

1)

A car of mass 5 tons moves uniformly on a horizontal road. If the resistance magnitude of motion equals 35 kg.wt. per ton of mass. find the force magnitude of the motor of the car.

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2)



A train of mass 300 tons is pulled by an engine with a constant force of a magnitude 810 kg.wt. under the action of a resistance whose magnitude is proportional to the square of its speed. If the maximum velocity of the train is equal to 30 m./sec., find the rate of the resistance per ton of the train's mass when its velocity is 90 km./h.

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3)



The total weight of a soldier and his equipments is 90 kg.wt. and the air resistance magnitude to his motion is proportional to the square of his speed. If the landing maximum velocity of the soldier is 12 km./h., find the air resistance magnitude when his velocity is 8 km./hr.

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4)

A car of mass 5 tons , if its engine is stopped , it moves down a road inclined to the horizontal at an angle θ where $\sin \theta = \frac{1}{100}$ with a uniform velocity. Calculate the resistance magnitude for each ton of its mass measured in kg.wt.

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5)

A locomotive of mass 80 tons pulls a train consisting of 5 carriages on a straight line with uniform velocity under the action of resistance of magnitude equals 8 kg.wt. per ton of mass. Find the mass of each carriage if the force magnitude of the engine = 2240 kg.wt. « 40 tons. »

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6)

An engine of mass 30 tons and force magnitude 51 ton.wt. pulls a number of train cars each of 10 tons to ascend a slope inclined at 30° to the horizontal with a uniform velocity. If the resistance magnitude of the motion of the engine and the cars is 10 kg.wt. per ton of the mass , find the number of cars.

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Lesson 6: Newton's second law

The rate of change of momentum with respect to the time is proportional to the acting force and takes place in the direction in which the force is acting.

∴ The mathematical expression of newton's second law is $\frac{d}{dt}(m\vec{v}) \propto \vec{F}$

∴ $\frac{d}{dt}(m\vec{v}) = k\vec{F}$ (where k is the positive constant of the proportion).

, then $\frac{d}{dt}(m\vec{v}) = m \frac{d\vec{v}}{dt} = m\vec{a}$

Thus the general mathematical form of newton's second law became $\frac{d}{dt}(m\vec{v}) = \vec{F}$

In the case of m is constant we get:

$m\vec{a} = \vec{F}$ (It is called the vector form of the equation of motion of the body)

or $ma = F$ (It is called the algebraic measure form of the equation of motion of the body)

Important remarks

(1) As using the law $F = m.a$, F should be in absolute units as follow :

$$F \text{ (newton)} = m \text{ (kg)} \times a \text{ (m./sec.}^2\text{)} \quad , \quad F \text{ (dyne)} = m \text{ (gm)} \times a \text{ (cm./sec.}^2\text{)}$$

For example :

If a body of mass 3 kg. moves with an acceleration as magnitude 5 m./sec.², then the force which acquired this acceleration is $F = m.a = 3 \times 5 = 15$ newtons.

(2) The weight of the body (W) in unit kg.wt. = (numerically) the magnitude of the mass of this body in unit kg.

For example :

- The body whose mass is 5 kg. , then its weight is 5 kg.wt. = $5 \times 9.8 = 49$ newton.
- The body whose mass is 10 gm. , then its weight is 10 gm.wt. = $10 \times 980 = 9800$ dyne.

Generally :

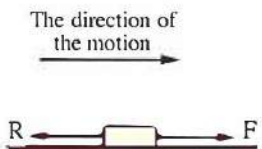
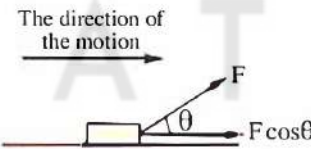
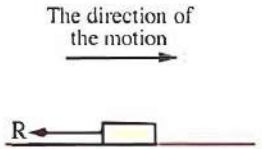
The body whose mass m , its weight $W = \begin{cases} m & \text{(gravitational unit)} \\ m.g & \text{(absolute units)} \end{cases}$

3) If we stopped the force or stopped the engine ,
then (the engine force) $F = \text{zero}$.

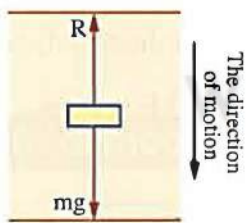
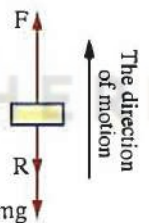
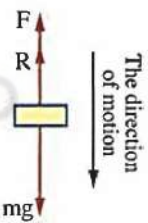
4) If a body moves with a uniform acceleration in a straight line, then:

- The resultant of the components of the acting forces on the body in the direction of motion of the body is equal to $m.a$
- The resultant of the components of the acting forces in the perpendicular direction of the motion of the body = zero.

The most common applications on the horizontal motion

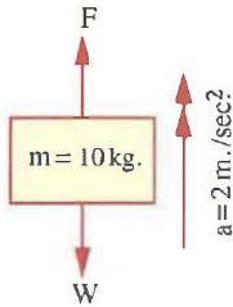
1. The motion of a body under action of a horizontal force ($\vec{F}$) and resistance of magnitude (R)  $F - R = m.a$	2. The motion of the body under action of a force inclines to the horizontal at an angle θ  $F \cos \theta = m.a$	3. The motion of a body as firing a bullet, using brakes or stopping the engine , then $F = 0$  $- R = m.a$
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The most common applications on the vertical motion

1. Falling a body vertically downwards inside sandy soil  $mg - R = m.a$	2. The moving of an aeroplane , balloon or aircraft vertical motion upwards  $F - R - mg = m.a$	3. The moving of an aeroplane , balloon or aircraft vertically downwards  $mg - R - F = m.a$
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In each of the following cases : The force $\vec{F}$ acts upon the body whose mass is m and acquires it a uniform acceleration shown in the figures , both magnitude and direction , find : F

1)



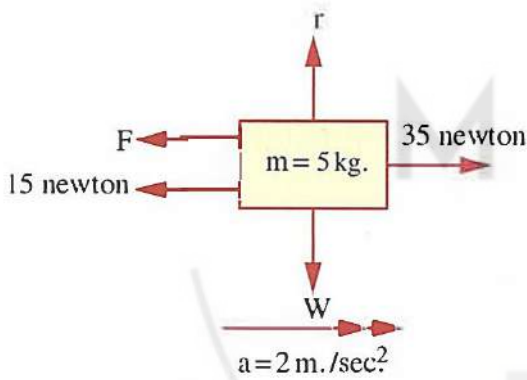
Sol.

$$F - W = ma$$

$$\therefore F - 10 \times 9.8 = 10 \times 2$$

$$\therefore F = 118 \text{ newton.}$$

2)



Sol.

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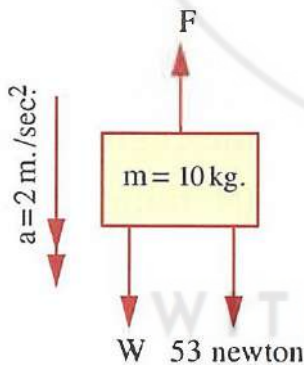
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3)



Sol.

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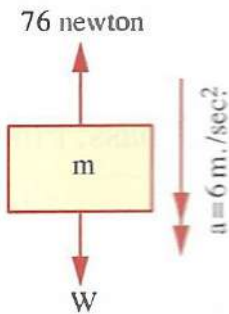
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📖 In each of the following cases : The force $\vec{F}$ acts upon the body whose mass is m and acquires it a uniform acceleration shown in the figures both magnitude and direction , find : m

1)

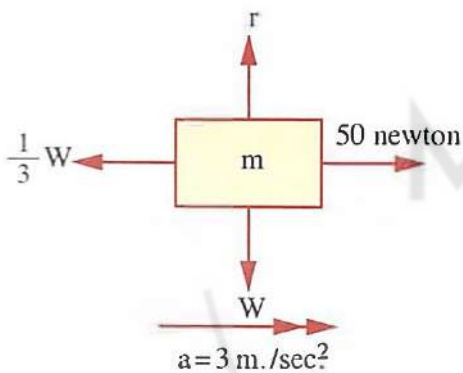


Sol.

$$m \times 9.8 - 76 = 6m$$

$$\therefore m = 20 \text{ kg.}$$

2)



Sol.

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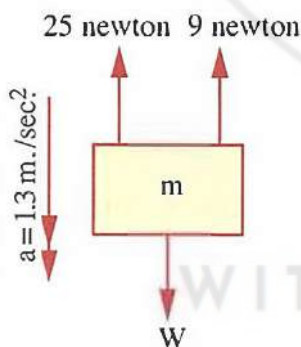
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3)



Sol.

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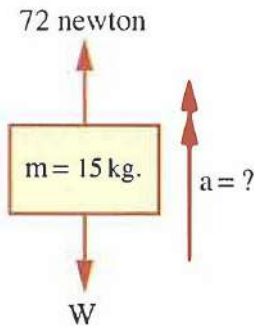
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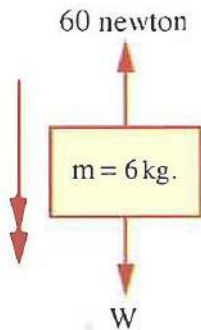
 In each of the following cases : The force $\vec{F}$ acts upon the body whose mass is m and acquires it a uniform acceleration a measured in m/sec^2 , find : a

1)



Sol. $72 - 15 \times 9.8 = 15 a$ $\therefore a = -5 \text{ m}/\text{sec}^2$

2)



Sol.

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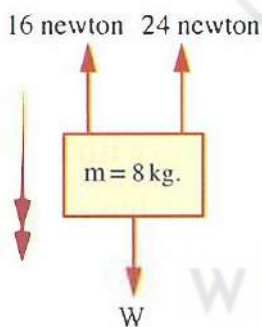
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3)



Sol.

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Exercises

[1] Choose the correct answer:

1)	 The weight magnitude of the body of mass 5 kg. is (a) $\frac{25}{49}$ N. (b) 5 N. (c) 49 N. (d) 49 kg.wt.	
2)	A body of mass 40 kg. , then its weight magnitude is (a) 40 N. (b) $\frac{200}{49}$ N. (c) 392 kg.wt. (d) 40 kg.wt.	
3)	The force that if acts on a body of mass 50 kg. causes it to accelerate 160 cm./sec^2 equals (a) 80 dyne. (b) 80 newton. (c) 8000 dyne. (d) 8000 newton.	
4)	$\frac{1}{7} \text{ gm.wt.} = \dots\dots\dots \text{ dyne.}$ (a) 1.4 (b) 14 (c) 140 (d) 1400	
5)	$9.8 \times 10^6 \text{ dyne} = \dots\dots\dots \text{ newton.}$ (a) 0.98 (b) 9.8 (c) 98 (d) 980	
6)	 A cyclist and the bike of mass 85 kg. moves with a uniform acceleration of magnitude 0.5 m./sec^2 , then the force that uses to cause this accelerations is (a) 42.5 kg.wt. (b) 42.5 newton. (c) 170 kg.wt. (d) 170 newton.	
7)	 A car is moving on a road without resistance with an acceleration of magnitude 1.47 m./sec^2 . If the engine force magnitude is 150 kg.wt. , then the mass of the car = kg. (a) 102 (b) 100 (c) 1000 (d) 220.5	

8)	A train moves with a speed of 72 km./h. and applies its brakes to stop. The train covers 250 m. to stop , then the resistance of the brakes per each ton of the mass = N. (a) 600 (b) 700 (c) 800 (d) 900
9)	A body of mass 5 kg. starts its motion from rest. If its speed increases by 3.92 m./sec. per second , then the magnitude of the force acts on the body equals kg.wt. (a) 19.6 (b) 15.2 (c) 2 (d) 1
10)	The magnitude of constant force needed to raise a body of mass 10 kg. from rest vertically upward 10 m. in 5 seconds is N. (a) 88 (b) 98 (c) 106 (d) 112
11)	A train moves in a straight line with a velocity , when the brakes are used the train will stopped after it covered a distance 250 meter , if the magnitude of the resistance = 800 Newton for each ton of the mass of the train , then its speed at the instant of using the brakes = m./sec. (a) 20 (b) 15 (c) 12 (d) 10
12)	A box of mass 100 kg. , is lifted off vertically upwards by a string with a uniform acceleration of magnitude 25 cm./sec ² , then the force of the tension magnitude in the string (neglect the resistance) = N. (a) 915 (b) 980 (c) 1 005 (d) 1 025
13)	A body of mass 150 kg. fell down from a point 140 cm. high above a sandy soil. It was imbedded in it. If the resistance force magnitude of the sandy soil was 2 250 kg.wt. , then the body imbedded in the sand distance cm. (a) 5 (b) 10 (c) 12 (d) 15
14)	A bullet of mass 98 gm. is fired horizontally from a pistol with speed 720 km./hr. at a wooden vertical barrier. It imbedded in it 10 cm. before it stopped , then the resistance magnitude of the wood to the bullet equals kg.wt. (a) 1 960 (b) 200 (c) 19 600 (d) 2 000

Answer the following questions:

1)



A mass of a magnitude 20 kg. is placed on a smooth horizontal plane. A horizontal force of a magnitude F acts on it to move it with a uniform acceleration of a magnitude 49 m./sec^2 . Find F

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2)



A car at rest of mass 4.9 tons , is acted upon by a force and its velocity gets 27 km./h. within one minute. Find the force magnitude acted upon the car in kg.wt.

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3)



If the force magnitude of the engine is equal to 2.5 ton.wt. and mass of the train and the engine is 200 tons and if the train starts to move from rest , find the velocity of the train after half a minute.

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4)

 A force of magnitude 10 newtons acts on a body at rest of mass 8 kg. to move it in its direction with a uniform acceleration. Calculate the distance traveled after 12 sec. and its velocity.

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5)

 Find the resistance force magnitude of the brakes to the motion of the train in newton. per each ton of its mass if the velocity is 72 km./h. and the brakes stop the train after it traveled 250 metres , find the time needed for doing that.

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6)

 Find the horizontal force magnitude by which an engine of a train of mass 245 tons is pulled to speed up its velocity into 18 km./h. after it covers a distance of one kilometre on a horizontal railroad if the resistance force magnitude is 4 kg.wt./ton.

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WITH SHERIF SAAD

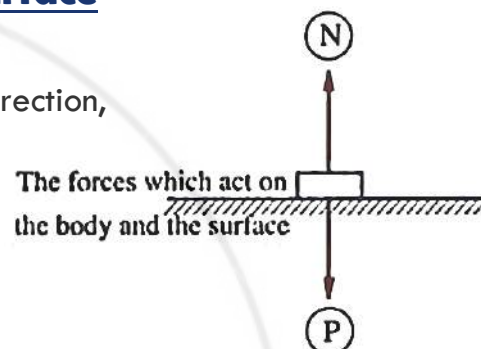
Lesson 7: Newton's third law

To every action, there is a reaction equal in magnitude and opposite in direction.

1) The force which acts on the body and the surface

They are two equal forces in magnitude and opposite in direction, they have one line of action, each of them acts a body

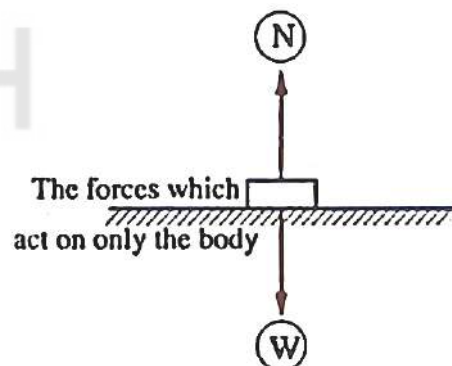
$$\therefore P=N$$



2) The force which acts only on the body

They are two forces equal in magnitude, opposite in directions and they have one line of action. Each of them acts on the body.

$$\therefore W=N$$



Applications on newton's laws (Motion of a body on a smooth inclined plane)

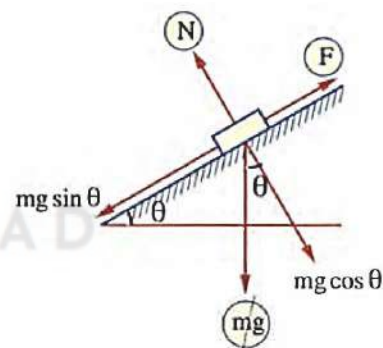
1. If $F > mg \sin \theta$, then the motion will be upwards with acceleration (a), then we get.

$$F - mg \sin \theta = ma$$

2. If $F < mg \sin \theta$, then the motion will be downwards with acceleration (a), then we get $mg \sin \theta - F = ma$

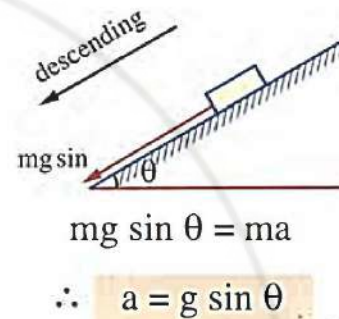
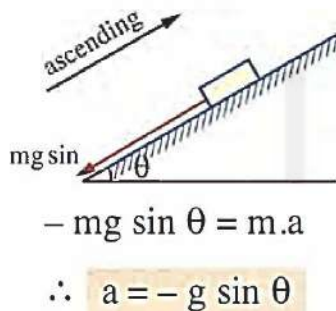
3. If $F = mg \sin \theta$, then the motion will be in uniform velocity.

i.e. $a = \text{zero}$



Remarks

- (1) In all the previous cases $N = mg \cos \theta$
- (2) As stopping the effect of the force, then $F = \text{zero}$
- (3) If the body moves with uniform velocity, then $a = \text{zero}$
- (4) If the body moves under the effect of its weight only on a smooth inclined plane, then



1)

A force of magnitude F acts on a body of mass 2.5 kg . placed on a smooth plane inclined to the horizontal at an angle θ where $\sin \theta = \frac{3}{5}$ in the direction of the line of greatest slope upwards. Find the magnitude of the acceleration of motion also find the reaction of the plane if :

(1) $F = 9.7 \text{ newton}$.

(2) $F = 1.5 \text{ kg.wt}$.

(3) $F = 22.2 \text{ newton}$.

Sol.

$$\therefore mg \sin \theta = \frac{5}{2} \times 9.8 \times \frac{3}{5} = 14.7 \text{ newton} = 1.5 \text{ kg.wt.}$$

(1) At $F = 9.7 \text{ newton}$:

$$\therefore F < mg \sin \theta$$

$\therefore$ The motion is downwards

$$\therefore mg \sin \theta - F = m.a$$

$$\therefore 14.7 - 9.7 = 2.5 \times a$$

$$\therefore 5 = 2.5 a$$

$$\therefore a = 2 \text{ m./sec}^2$$

(2) At $F = 1.5 \text{ kg.wt.} : \therefore F = mg \sin \theta$

$\therefore$ The body is in rest or does not change its state.

$$\therefore a = \text{zero}$$

(3) At $F = 22.2 \text{ newton}$:

$$\therefore F - mg \sin \theta = ma$$

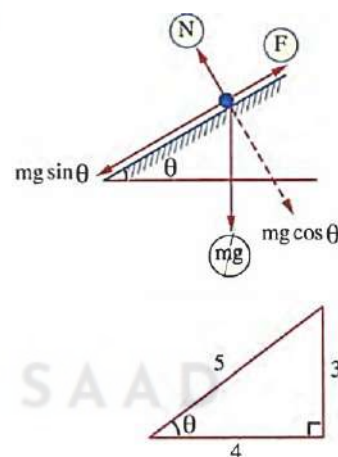
$$\therefore 22.2 - 14.7 = 2.5 a$$

$$\therefore F > mg \sin \theta$$

$$\therefore a = 3 \text{ m./sec}^2$$

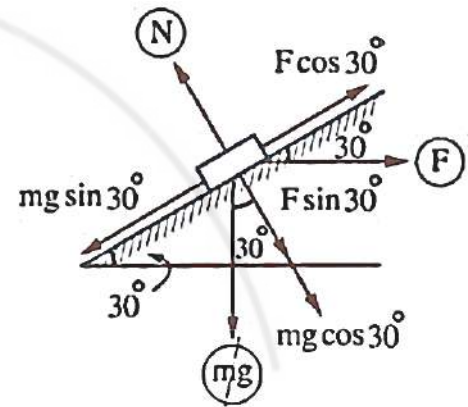
$\therefore$ The motion is upwards of the plane.

$$\therefore N = mg \cos \theta = \frac{5}{2} \times 9.8 \times \frac{4}{5} = 19.6 \text{ newton} = 2 \text{ kg.wt.}$$



2)

A body of mass 6 kg. is placed on an inclined smooth plane making an angle of measure 30° with the horizontal, then a horizontal force of magnitude $10\sqrt{3}$ kg.wt. acted on the body, its line of action lies in the vertical plane containing the line of greatest slope. Find the magnitude and the direction of the acquired acceleration also find the reaction of the plane.

Sol.

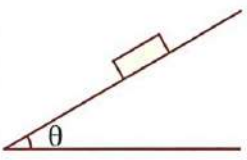
3)

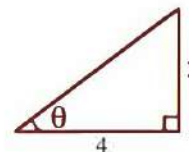
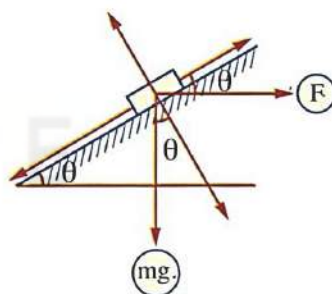
A body of mass 4 kg. is placed on a smooth inclined plane, then it slides from rest under effect of its weight only, to cover a distance 88.2 metre during 6 seconds. Find the measure of the inclination angle of this plane. If this body is projected on the same plane upwards in the direction of the line of the greatest slope with velocity 24.5 m/sec. Find the maximum distance that the body will move on this plane.

Sol.

Exercises

[1] Choose the correct answer:

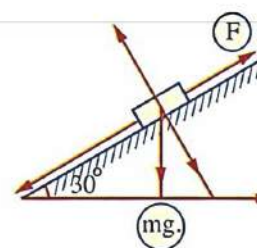
1)	 If a body moves on a smooth inclined plane and inclines with an angle of measure θ under the action of its weight only , then its acceleration of motion is equal to (a) g (b) $g \cos \theta$ (c) $g \sin \theta$ (d) zero.	
2)	 If a body moves on a smooth inclined plane under the action of its weight only , then its acceleration is only based on (a) its mass. (b) its weight. (c) the angle of inclination of the plane. (d) the reaction of the plane.	
3)	 In the opposite figure : The mass of the body placed on a smooth plane is $m = 12 \text{ kg}$. It starts to move from rest under the action of the force $\vec{F}$ whose magnitude is 8 kg.wt. Complete : First : The magnitude of the acceleration of motion = m./sec^2 (a) 9.8 (b) 4.9 (c) $\frac{49}{75}$ (d) $\frac{49}{25}$ Second : The distance which the body travels on the plane in 3 seconds from the beginning of the motion is metres. (a) 9.8 (b) 2.94 (c) 3.5 (d) 4.9 Third : The magnitude of reaction of the plane = kg.wt. (a) 14.4 (b) 7.2 (c) 28.8 (d) 3.6	



4)

 In the opposite figure :

If the mass of the body placed on the smooth plane is 2 kg. and the body starts to move from rest under the action of the force $\vec{F}$ whose magnitude is 1.5 kg.wt. , then



First : The acceleration of motion equals

- (a) 2.45 m./sec² down the plane. (b) 2.45 m./sec² up the plane.
(c) 4.9 m./sec² down the plane. (d) 4.9 m./sec² up the plane.

Second : The speed of the body after 4 sec. = m./sec.

- (a) 9.8 (b) 4.9 (c) 2.45 (d) 1.96

Third : The magnitude of reaction of the plane = kg.wt.

- (a) 9.8 (b) 1 (c) $\sqrt{2}$ (d) $\sqrt{3}$

Answer the following questions:

1)

 A body of mass 1 kg. is placed on a smooth plane inclined to the horizontal at an angle of measure 30°. A force of magnitude 10 newton acts on the body along the line of the greatest slope upwards. Find the magnitude of the force of reaction of the plane on the body. Find also the magnitude of its acceleration.

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2)

 A body of mass 10 kg. is placed on an inclined smooth plane with an angle of sine = $\frac{3}{5}$. If a force of 80 newton affects up on the body in the direction of the line of the greatest slope. Then find the magnitude and direction of the acceleration and the reaction of the plane.

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3)

 A body of mass 12 kg is placed on a smooth plane inclines at 30° , to the horizontal. A force of magnitude 88.8 newton acts in the direction of the line of the greatest slope upwards the plane. Find the velocity of this body after 14 seconds from the beginning of the motion. If the force acting on the body is ceased at this moment, find the distance which the body moves on the plane after that until it is at rest instantaneously.

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4)

 A body of mass 30 kg. moves upwards a smooth inclined plane inclined at 30° to the horizontal under the action of a force of magnitude F newtons in the direction of the line of the greatest slope upwards with an acceleration of magnitude 1.5 m./sec^2 . and if the force is decreased to its half find the acceleration by which this body moves on the same.

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5)

 A body of mass 25 kg. is placed on a smooth plane inclines with an angle of measure θ where $\tan \theta = \frac{4}{3}$ A horizontal force of magnitude 30 kg.wt., acts in the direction of the plane and its line of action lies in the vertical plane passing through the line of the greatest slope to the plane. Find the generated acceleration and the magnitude of the reaction force of the plane.

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Lesson 8: Follow applications on Newton's law

1)

A body of mass 10 kg. is placed on a horizontal rough plane and the coefficient of static friction between the body and the plane is 0.4 and the coefficient of kinetic friction between the body and the plane is 0.3 , the magnitude of a horizontal force acts on the body increases till the body moved. Find the magnitude of the friction force between the body and the plane in the following cases :

(1) The body about to move.

(2) During the motion.

(3) Before motion.

Sol.

(1) When the body about to move

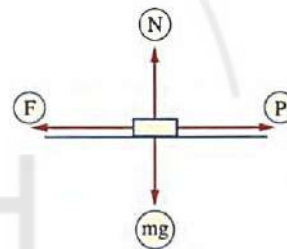
(use the limiting static friction)

$$F_s = \mu_s \times N = \mu_s \times mg = 0.4 \times 10 \times 9.8 = 39.2 \text{ newton.}$$

(2) During the motion (use the kinetic friction)

$$F_K = \mu_K \times N = \mu_K \times mg = 0.3 \times 10 \times 9.8 = 29.4 \text{ newton.}$$

(3) Before motion (use the static friction $F \leq F_s$) and F is always equals the magnitude of the horizontal force (P) as long as the body did not start its motion.



2)

A body of mass 140 gm. is placed on a rough horizontal plane , then the body is pulled by a horizontal force of magnitude 49000 dyne , which moves it with acceleration of magnitude 105 cm./sec². Find the coefficient of the kinetic friction between the body and the plane.

Sol.

∴ The vertical forces are in equilibrium

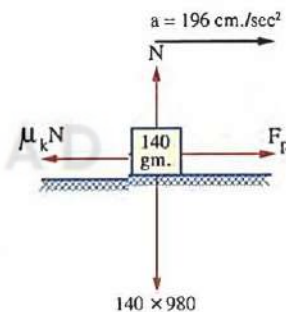
$$\therefore N = 140 \times 980 \text{ dyne}$$

The motion's equation of the body is :

$$F_p - \mu_K N = ma \quad \therefore 49000 - 140 \times 980 \mu_K = 140 \times 105$$

$$\therefore \mu_K = \frac{1}{4}$$

∴ The coefficient of the kinetic friction between the body and the plane = $\frac{1}{4}$



- 3) A wooden box of mass 2 kg. is placed on the top of a rough inclined plane of length 50 cm. and of height 30 cm. , then the box slid till it reached to the bottom of the plane after $\frac{5}{7}$ second. Find :
- (1) The magnitude of the acceleration of the motion of the box.
 - (2) The coefficient of the kinetic friction between the box and the plane.
 - (3) The magnitude of the kinetic friction force between the box and the plane measured in newton.

Sol.

$$\therefore s = v_0 t + \frac{1}{2} a t^2$$

$$\therefore 50 = \text{zero} + \frac{1}{2} \times a \times \left(\frac{5}{7}\right)^2$$

$$\therefore a = 196 \text{ cm./sec}^2 = 1.96 \text{ m./sec}^2$$

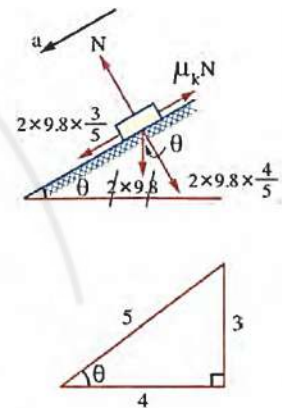
$$\therefore \text{The equation of motion of the box is : } 2 \times 9.8 \times \frac{3}{5} - \mu_K N = ma$$

$$\therefore N = 2 \times 9.8 \times \frac{4}{5}$$

$$\therefore 2 \times 9.8 \times \frac{3}{5} - 2 \times 9.8 \times \frac{4}{5} \mu_K = 2 \times 1.96$$

$$\therefore \mu_K = \frac{1}{2}$$

$$\therefore \text{The magnitude of the kinetic friction force} = \mu_K N = \frac{1}{2} \times 2 \times 9.8 \times \frac{4}{5} = 7.84 \text{ newton.}$$



- 4) A rough inclined plane is of length 250 cm. and height 150 cm. A body at the bottom of the plane is projected upwards the plane. Find the least velocity with which the body should be projected to reach the top of the inclined plane.
Given that the coefficient of the kinetic friction between the body and the plane = $\frac{1}{2}$

Sol.

$\therefore$ The height of the plane = 150 cm. and its length = 250 cm.

$$\therefore \sin \theta = \frac{3}{5}, \cos \theta = \frac{4}{5}$$

$\therefore$ The body is projected upwards

$$\therefore \text{The equation of motion is } -mg \sin \theta - \mu_K N = ma$$

$$\therefore N = mg \cos \theta \quad \therefore -mg \sin \theta - \mu_K mg \cos \theta = ma$$

Dividing by m :

$$\therefore a = -g \sin \theta - \mu_K g \cos \theta = -980 \times \frac{3}{5} - \frac{1}{2} \times 980 \times \frac{4}{5} = -980 \text{ cm./sec}^2$$

To find the velocity of projection (v_0)

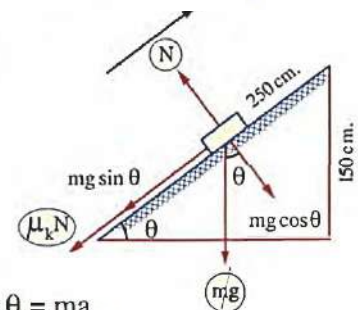
Consider $v = \text{zero}$ after covering a distance 250 cm. with acceleration = -980 cm./sec^2

$$\therefore v^2 = v_0^2 + 2as$$

$$\therefore \text{zero} = v_0^2 - 2 \times 980 \times 250$$

$$\therefore v_0^2 = 490000$$

$$\therefore v_0 = 700 \text{ cm./sec.} = 7 \text{ m./sec.}$$



Remarks

(1) Be at rest : In this case we find that $mg \sin \theta < \mu_s N$

$$\therefore N = mg \cos \theta, \mu_s = \tan \lambda$$

$$\therefore mg \sin \theta < mg \cos \theta \tan \lambda$$

$$\therefore \tan \theta < \tan \lambda$$

$$\therefore \boxed{\theta < \lambda}$$

i.e. The measure of the angle of inclination of the inclined plane (θ) is smaller than the measure of the angle of static friction (λ)

(2) Be at rest but it is about to move :

$$\text{In this case we find that : } mg \sin \theta = \mu_s N$$

$$\text{i.e. } mg \sin \theta = \mu_s \times mg \cos \theta$$

$$\therefore \boxed{\theta = \lambda}$$

(3) Be at rest instantaneously then begins turning back downwards the plane.

$$\text{In this case we find that : } mg \sin \theta > \mu_s N$$

$$\text{i.e. } \theta > \lambda$$

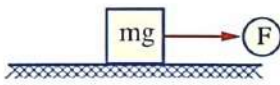
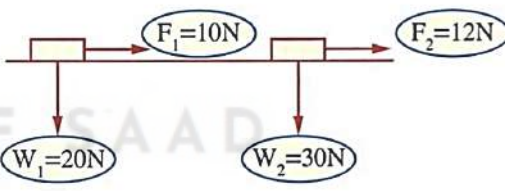
- To distinguish between the three cases this requires from us.

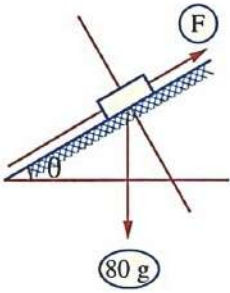
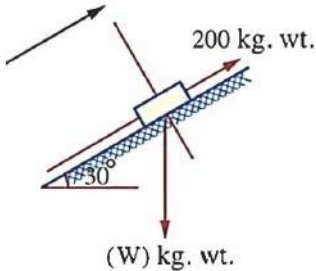
Either compare between the magnitude of the limiting static friction force ($\mu_s N$) and the magnitude of ($mg \sin \theta$)

Or compare between the measure of the static friction angle (λ) and the inclination's angle (θ) of the plane.

Exercises

[1] Choose the correct answer:

- 1) If a body of mass m is placed at the top of an inclined rough plane makes with the horizontal an angle of measure θ and the coefficient of kinetic friction between the body and the plane is μ_k . If the body slides under action of its weight only , then the acceleration =
- (a) $g \sin \theta$ (b) $g \cos \theta$
(c) $g (\mu_k \sin \theta - \cos \theta)$ (d) $g (\sin \theta - \mu_k \cos \theta)$
- 2) A body of mass m is projected upon a rough plane inclined to the horizontal at an angle of measure θ . If the body reaches to the maximum distance on the plane in t_1 then it returns to the projection point in t_2 , then
- (a) $t_1 > t_2$ (b) $t_1 < t_2$ (c) $t_1 = t_2$
(d) the comparison depends on the ratio between the friction force and the body weight
- 3) **In the opposite figure :**
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- A body of mass m is placed on a horizontal of rough plane , the coefficient of friction between the body and the plane equals $\frac{1}{3}$. The body moves with uniform speed under action of force of magnitude F . If a part of mass 10 kg. is separated from the body then to preserve the uniform motion the force should be reduced by kg.wt.
- (a) 10 (b) 30 (c) $\frac{98}{3}$ (d) $\frac{10}{3}$
- 4) **In the figure shown :**
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- Two bodies of the same material of weights 20 and 30 newton are placed on a horizontal rough plane. Two horizontal forces of magnitude 10 and 12 newton act on them respectively , the first body becomes about to move , while the other moves with uniform velocity , then the ratio between the coefficient of static friction and the coefficient of kinetic friction is
- (a) 3 : 2 (b) 4 : 3 (c) 5 : 4 (d) 6 : 5

5)	In the opposite figure : A body of mass 80 kg. is placed on a rough plane inclined to the horizontal at an angle $\sin^{-1} \frac{3}{5}$ and the coefficient of kinetic friction between them is $\frac{1}{4}$. Force of magnitude 100 kg.wt. acts on the body and acts in direction of line of the greatest slope upward for 4 seconds then the force is ceased and the body comes to instantaneous rest after t seconds , then t = sec. (a) $\frac{9}{2}$ (b) $\frac{9}{4}$ (c) $\frac{3}{2}$ (d) $\frac{5}{4}$	
6)	A car of mass 2 tons moves upwards along the line of the greatest slope of a plane inclined at an angle of sine = $\frac{1}{20}$ to the horizontal against a resistance of magnitude 40 kg.wt. for each ton of its mass. If the car is starting from rest, and covers 4.9 metres in 10 sec. , then the force magnitude of its engine = kg.wt. (a) 100 (b) 150 (c) 200 (d) 250	
7)	In the opposite figure : A body of weight (W) kg.wt. is placed on a rough inclined plane inclines to the horizontal by an angle of measure 30°. A force of magnitude 200 kg.wt. acts on the body in the direction of the line of the greatest slope of the plane upwards to move it upwards the plane with acceleration 0.98 m/sec^2 against a resistance of magnitude 784 newton , then W = kg.wt. (a) 200 (b) 1960 (c) 20 (d) 196	

Answer the following questions:

1)

 A body of weight 10 kg.wt. is placed on a rough horizontal plane. A force of magnitude 37 newtons acts on this body to move it on the horizontal plane with a uniform acceleration of magnitude $\frac{5}{4}$ m./sec.², find the coefficient of the kinetic friction between the body and the plane.

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2)

 A body of mass 2 kg. is placed on a rough horizontal plane and the coefficient of the kinetic friction between the body and the plane is $\frac{1}{2}$, find magnitude of the horizontal force making the body move with a uniform acceleration a where :

(1) $a = 5$ m./sec.²

(2) $a = 1$ m./sec.²

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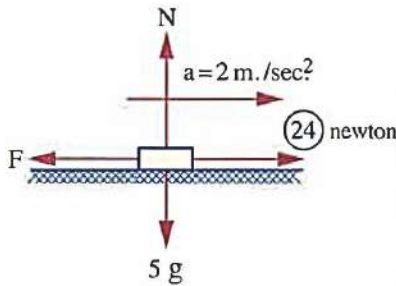
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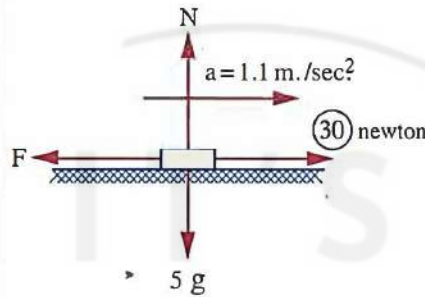
3)

In each of the following figures, a body of mass 5 kg, is placed on a rough plane and the coefficient of the kinetic friction between the body and the plane μ_K , calculate μ_K in each case, F is magnitude of the friction force:

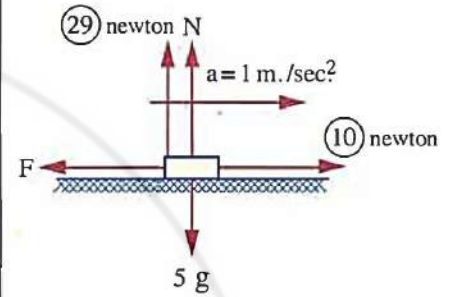
(1)



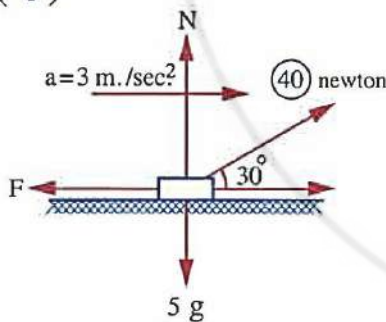
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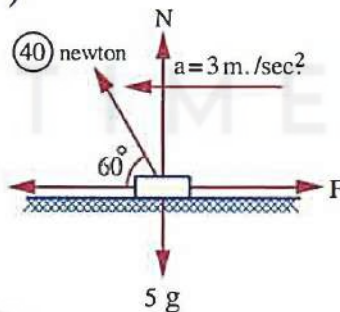
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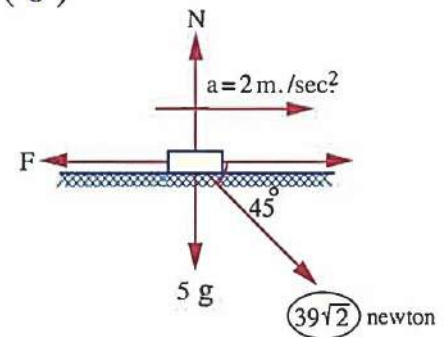
(4)



(5)

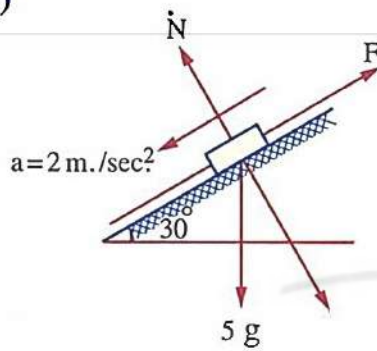


(6)

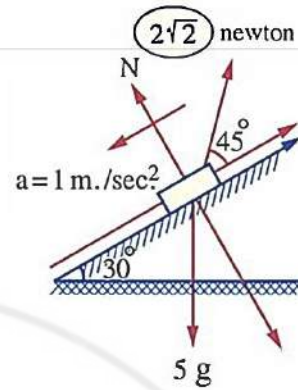


WITH SHERIF SAAD

(7)



(8)



4)

 A rough inclined plane of length 250 cm. and height 150 cm. , a body is placed on it to slide downwards the plane and the acceleration of motion is equal to 196 cm./sec^2 . Find the coefficient of the kinetic friction , then find the velocity of the body in m./sec. after it travels (cuts) 200 cm. on the plane.

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WITH SHERIF SAAD

5)

 A body of mass 2 kg. is placed on a rough plane which inclines at 30° to the horizontal. A horizontal force of magnitude 20 newtons acts on the body towards the plane, then the body moves in a uniform velocity. Find the coefficient of the kinetic friction between the body and the plane.

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6)

 A body of mass 1 ton is needed to drag into a rough plane inclines with an angle of measure θ where $\tan \theta = \frac{3}{4}$ to the horizontal by a force parallel to the plane in the direction of the line of the greatest slope upwards. Find the coefficient of the kinetic friction between the body and the plane if the least force preserves the body moving up on the plane is of magnitude 1400 kg.wt.

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